A Novel Approach to Extra Dimensions

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Abstract

Four-dimensional spacetime, together with a natural generalisation to extra dimensions, is obtained through an analysis of the structures and symmetries deriving from possible arithmetic expressions for one-dimensional time. On taking the infinitesimal limit this simple one-dimensional structure can be consistently equated with a homogeneous form of arbitrary dimension possessing both spacetime and more general symmetries. An extended 4-dimensional manifold, with the associated spacetime symmetry, provides a natural breaking mechanism for a higher-dimensional form and symmetry of time. It will be described how this symmetry breaking leads to a series of distinct properties of the Standard Model of particle physics, deriving directly from the natural mathematical development of the theory.

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1 Introduction

In 4-dimensional spacetime an interval of proper time ds can be expressed locally in terms of the Minkowski metric $\eta = \mathrm{diag}(+1, -1, -1, -1)$ for suitable local coordinates x^a as:

$$ds^2 = \eta_{ab} dx^a dx^b \tag{1}$$

with $a, b \in \{0, 1, 2, 3\}$. A typical approach to extra dimensions would involve an extension to the range of the indices a, b in the above quadratic expression, with the metric correspondingly replaced by an $n \times n$ matrix for the n-dimensional spacetime

generalisation. An extended 4-dimensional spacetime manifold, with the local metric of equation 1, might be identified for example through the spontaneous compactification of the extra dimensions, in principle resulting in residual physical properties that might be observable in 4-dimensional spacetime.

For the theory described in this paper in place of generalising the right-hand side of equation 1 for a higher-dimensional *spacetime* structure we consider a generalisation of the overall expression as constrained by the *time* interval on the left-hand side. For example from the perspective of the linear one-dimensional flow of time it is equally permitted to write the cubic expression:

$$ds^3 = \alpha_{abc} dx^a dx^b dx^c \tag{2}$$

with each coefficient $\alpha_{abc} \in \{-1, 0, 1\}$ for $a, b, c \in \{1 \dots n\}$ for the n-dimensional case. In the following section we describe how to express the general form of this type of generalisation. The means of identifying the 4-dimensional structure of equation 1 within this generalisation will be described in section 3 and will provide the means of breaking the full symmetry of expressions such as equation 2.

In sections 4–6 it will be described how natural mathematical extensions of this idea lead directly to the identification of physical properties such as fractional charges for quark states and a left-right asymmetry of a kind closely resembling the structure of the Standard Model. This development leads to the consideration of an E_8 symmetry in section 7 as a final stage in this progression that might in principle accommodate the full set of Standard Model properties.

2 Time and Spacetime

In this paper it is described how consideration of a multi-dimensional form of time and symmetry breaking over a 4-dimensional spacetime manifold leads directly to structures which exhibit a close resemblance to the Standard Model. We begin here by developing the underlying idea. A finite interval of time represented by the real number $s \in \mathbb{R}$ can be algebraically expressed in terms of other real numbers x^a $(a = 1, 2, 3 \dots)$ in an endless variety of ways, for example we can have $s = x^1(x^2x^3 + x^4)$ and so on simply by employing the basic arithmetic structure of the real line.

The broad range of possible expressions for a finite interval s in terms of an arbitrary number of variables $\{x^a\}$, $a = 1 \dots n$, will be constrained to a more restrictive structure in the limit of infinitesimally small temporal intervals. We first consider this limit for the trivial case with the flow of time s expressed in terms of a single real variable x^1 only for which we have simply $s = x^1$. This can symbolically be written as $\delta s = \delta x^1$ as we approach the limit of infinitesimal intervals. We then express the rate of change of x with respect to s in this limit as:

$$v^1 = \frac{dx^1}{ds} \equiv \left.\frac{\delta x^1}{\delta s}\right|_{\delta s \to 0} = 1 \tag{3}$$

For the case with multiple real numbers $\{x^a\} \in \mathbb{R}^n$ representing the flow of time s each will be associated with a corresponding rate of change $v^a = dx^a/ds$ with respect

to time. For example, we may consider the propagation of time expressed for an infinitesimal interval as:

$$\begin{aligned}(\delta s)^2 &= (\delta x^1)^2 + (\delta x^2)^2 + (\delta x^3)^2 && (4)\\ &= \eta_{ab}\delta x^a \delta x^b \qquad \text{with} \quad \eta = \mathrm{diag}(+1,+1,+1) && (5)\end{aligned}$$

where $a, b \in \{1, 2, 3\}$ (and with the conventional summation over repeated indices implied throughout this paper). Dividing by $(\delta s)^2$ and taking the limit $\delta s \to 0$ this can be written as $\eta_{ab}v^a v^b = 1$ or $(v^1)^2+(v^2)^2+(v^3)^2 = 1$, which is invariant under the group, O(3), of orthogonal transformations in three dimensions applied to $\boldsymbol{v}_3 = (v^1, v^2, v^3) \in \mathbb{R}^3$. The question is then how to express the general case for the composition and symmetries of a multi-dimensional set of velocities $\{v^a\} \in \mathbb{R}^n$.

The infinitesimal elements of time can be written most generally, taking care to balance the order of the vanishing elements in each term, as:

$$\delta s = \alpha_a \delta x^a + \sqrt{\alpha_{bc}\delta x^b \delta x^c} + \sqrt[3]{\alpha_{def}\delta x^d \delta x^e \delta x^f} + \ldots\ldots \qquad (6)$$

Here the coefficients $\alpha_{abc...}$ are each equal to ± 1 or 0 since we wish to express the δs purely in terms of simple arithmetic relations of the δx^a. In equation 6 each term divides δs into a separate portion of time:

$$\delta s \quad = \quad \delta s_1 \quad + \quad \delta s_2 \quad + \quad \delta s_3 \quad + \quad \ldots \qquad (7)$$

where each term δs_p is the p^{th}-root of a homogeneous polynomial of order p in the $\{\delta x^a\}$. Taking each term in turn, dividing by the interval δs_p in each case and taking the limit $\{\delta s_p, \delta x^a\} \to 0$ we find:

$$\begin{aligned} & \delta s_p &=& \sqrt[p]{\alpha_{abc...}\delta x^a \delta x^b \delta x^c \ldots} && (8)\\ \text{divide by } \delta s_p: \quad & 1 &=& \sqrt[p]{\alpha_{abc...}v^a v^b v^c \ldots} && (9)\\ \text{that is:} \quad & \alpha_{abc...}v^a v^b v^c \ldots &=& 1 && (10)\\ \text{which we write:} \quad & L(\boldsymbol{v}) &=& 1 && (11)\end{aligned}$$

where L is a homogeneous polynomial of order p in the components v^a; it can be considered as a map from the elements of a real n-dimensional vector space $\boldsymbol{v} \in \mathbb{R}^n$ onto the unit $1 \in \mathbb{R}$. Equation 11 is taken to express the general mathematical form of multi-dimensional temporal flow and it is the central equation of this paper. The symmetries of $L(\boldsymbol{v}) = 1$ will be represented by groups acting on the vector space $\mathbb{R}^n$ such that for all elements g of the group G and all vectors $\boldsymbol{v} \in \mathbb{R}^n$ satisfying $L(\boldsymbol{v}) = 1$ we have $L(\sigma_g(\boldsymbol{v})) = L(\boldsymbol{v}') = 1$ where $\sigma_g(\boldsymbol{v})$ represents the action of the group element $g \in G$ on the vector $\boldsymbol{v} \in \mathbb{R}^n$.

Quadratic forms in general, including the 4-dimensional form:

$$L(\boldsymbol{v}_4) = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = 1 \qquad (12)$$

that is $L(\boldsymbol{v}_4) = \eta_{ab}v^a v^b = 1$ for $a, b \in \{0, 1, 2, 3\}$ with the Minkowski metric $\eta = \mathrm{diag}(+1, -1, -1, -1)$, $\boldsymbol{v}_4 \in \mathbb{R}^4$ and with Lorentz $SO^+(1,3)$ symmetry, and the norm of an element of a division algebra ($\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$), together with their symmetry groups,

are expected to be particularly significant forms of $L(\boldsymbol{v}) = 1$. This is due to their close relation to Clifford algebras and Euclidean spatial geometry, describing the geometry of external space and spacetime.

Equation 12 is the special case of equation 10 in the form of the particular case of equation 1 described in the introduction. The generalisation to the cubic form in equation 2 is incorporated within the general expressions of equations 10 and 11. Other possible forms of $L(\boldsymbol{v}) = 1$ include the determinants of matrices, which are homogeneous polynomials in the matrix elements.

With various different forms of progression in time to be considered, in general the subscript n in the notation $L(\boldsymbol{v}_n) = 1$ indicates collectively the vector space $\mathbb{R}^n$, the implied form L and the corresponding symmetry group G (respectively $\boldsymbol{v}_4 \in \mathbb{R}^4$, $L(\boldsymbol{v}_4) = \eta_{ab}v^a v^b = 1$ and $G = \mathrm{SO}^+(1,3)$ in the above example for $n = 4$). The notation $L(\hat{\boldsymbol{v}}) = 1$ and $\hat{G}$, with a 'hat' above a kernel symbol, will denote the highest-dimensional form of time considered and its symmetry respectively.

Given a possible n-dimensional form of progression in time, $L(\boldsymbol{v}_n) = 1$, the vector $\boldsymbol{v}_n \in \mathbb{R}^n$ may be written as the ordered set of velocities:

$$\begin{aligned} \boldsymbol{v}_n &= \{\, v^1, \quad v^2, \ldots \quad v^n \,\} && (13) \\ &= \left\{ \frac{dx^1}{ds}, \frac{dx^2}{ds}, \ldots \frac{dx^n}{ds} \right\} && (14) \end{aligned}$$

the values of which are unchanged by a numerical translation of the real variables,

$$x^a \to x^a + r^a \tag{15}$$

for any constant set $\{r^a\} = \boldsymbol{r}_n \in \mathbb{R}^n$, or for a subset of $\mathbb{R}^n$. Above we described a possible symmetry of $L(\boldsymbol{v}) = 1$ with the action of a group G mixing the numerical components v^a, which represent elements of the flow of time dx^a/ds. Here we have a further symmetry implicit in $L(\boldsymbol{v}) = 1$ with respect to translations of the numerical variables as $x^a \to x^a + r^a$. That is, we also have trivially:

$$\boldsymbol{v}_n = \left\{ \frac{d(x^1 + r^1)}{ds}, \frac{d(x^2 + r^2)}{ds}, \ldots \frac{d(x^n + r^n)}{ds} \right\}. \tag{16}$$

satisfying $L(\boldsymbol{v}_n) = 1$. The relation between the 'translation symmetry' of $L(\boldsymbol{v}) = 1$ and the 'rotation symmetry', more generally denoted by the action $\sigma_g(\boldsymbol{v})$ for $g \in G$, is key to the development of the geometrical structure of the theory.

3 Extra Dimensions

We initially consider $\hat{G} = \mathrm{SO}^+(1,9)$ to be provisionally taken as the full symmetry group for the form $L(\hat{\boldsymbol{v}}) = L(\boldsymbol{v}_{10}) = 1$, which in turn is the 10-dimensional extension of equation 12 (this is also equivalent to an extension for extra spacetime dimensions as described for equation 1). Here an extended base manifold M_4 arises through employing for four of the ten translational degrees of freedom of $L(\boldsymbol{v}_{10}) = 1$, in the manner described in equation 16. For this model we hence obtain the structures described in figure 1.

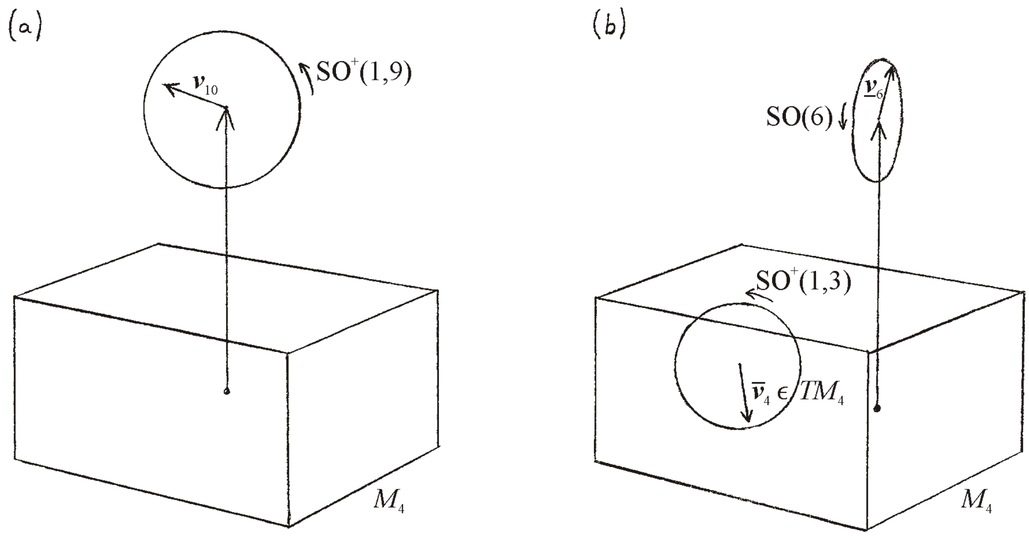

Figure 1: (a) The full symmetry group $\hat{G} = \mathrm{SO}^+(1,9)$ over the base space M_4 (b) broken to the internal symmetry SO(6) with external subgroup $\mathrm{SO}^+(1,3) \subset \mathrm{SO}^+(1,9)$ acting on the tangent space TM_4. (An underline, as for $\underline{\boldsymbol{v}}_6$, or an overline, as for $\overline{\boldsymbol{v}}_4$, may be used for internal and external objects respectively, in case of ambiguity).

This figure also represents the manner in which the identification of an extended 4-dimensional background manifold breaks the symmetry of the higher-dimensional form of time $L(\hat{\boldsymbol{v}}) = 1$. However while quadratic spacetime forms such as $L(\boldsymbol{v}_{10}) = 1$ are included as a possible structure of equation 11 more generally cubic or higher-order polynomial forms of time are also permitted (as initially described for equation 2).

We approach this generalisation via the group $\mathrm{SL}(2,\mathbb{C})$ as the two-to-one cover of $\mathrm{SO}^+(1,3)$, which may be exhibited by mapping a Lorentz vector $\boldsymbol{v}_4 \in \mathbb{R}^{1,3}$ into the space of 2×2 complex Hermitian matrices as:

$$\boldsymbol{v}_4 = (v^0, v^1, v^2, v^3) \;\rightarrow\; \boldsymbol{h}_2 = \boldsymbol{v}_4{\cdot}\boldsymbol{\sigma} = \begin{pmatrix} v^0+v^3 & v^1-v^2i \\ v^1+v^2i & v^0-v^3 \end{pmatrix} \subset \mathrm{h}_2\mathbb{C} \qquad (17)$$

where $\boldsymbol{\sigma}$ denotes the 2×2 identity matrix σ^0 together with the three Pauli matrices σ^a, that is $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. While the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ acts on the space $\mathbb{C}^2$, the group action for elements $S \in \mathrm{SL}(2,\mathbb{C})$ on the space $\mathrm{h}_2\mathbb{C}$ provides another representation given by:

$$\boldsymbol{h}_2 \rightarrow \boldsymbol{h}_2' = S\,\boldsymbol{h}_2\,S^\dagger \qquad (18)$$

This maps $\boldsymbol{h}_2 \rightarrow \boldsymbol{h}_2'$ onto a new 2×2 complex Hermitian matrix with the same determinant; hence mapping the components $v_4^a \rightarrow v_4'^a$ according to a Lorentz transformation of the real 4-vector $\boldsymbol{v}_4 \in \mathbb{R}^{1,3}$. This $\mathrm{SL}(2,\mathbb{C})$ action expresses the symmetry of $L(\boldsymbol{v}_4) = 1$ of equation 12 in a manner that naturally extends, consistent with equation 11, to an $\mathrm{SL}(3,\mathbb{C})$ symmetry of the cubic polynomial form $L(\boldsymbol{v}_9) = \det(\boldsymbol{v}_9) = 1$ with $\boldsymbol{v}_9 \in \mathrm{h}_3\mathbb{C}$. For this latter case, in identifying the base space M_4 through the translation symmetry of subspace of external vectors $\overline{\boldsymbol{v}}_4 \equiv \boldsymbol{h}_2 \in \mathrm{h}_2\mathbb{C} \subset \mathrm{h}_3\mathbb{C}$, the symmetry

is broken to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1) \subset \mathrm{SL}(3,\mathbb{C})$. The action of the external symmetry $\mathrm{SL}(2,\mathbb{C})$ on the full space $\mathrm{h}_3\mathbb{C}$ may then be considered. The 2×2 matrices $S \in \mathrm{SL}(2,\mathbb{C})$ can be embedded in 3×3 matrices acting on $\boldsymbol{v}_9 \in \mathrm{h}_3\mathbb{C}$ as:

$$\boldsymbol{v}_9 \;\rightarrow\; \left(\begin{array}{c|c} S & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} \boldsymbol{h}_2 & \psi_L \\ \hline \psi_L^\dagger & n \end{array}\right) \left(\begin{array}{c|c} S^\dagger & 0 \\ \hline 0 & 1 \end{array}\right) \tag{19}$$

This combines the vector representation of $\mathrm{SL}(2,\mathbb{C})$ on $\boldsymbol{h}_2 \in \mathrm{h}_2\mathbb{C}$ and the spinor representation on $\psi_L \in \mathbb{C}^2$ (taken to be left-handed), together with the scalar denoted $n \in \mathbb{R}$ (in line with the notation used for equation 20 below), in a single symmetry transformation which preserves $L(\boldsymbol{v}_9) = \det(\boldsymbol{v}_9) = 1$.

Hence by considering a higher-dimensional *cubic* form of time we have identified components of the 'extra dimensions' transforming as a spinor, namely the ψ_L of equation 19, which also transforms non-trivially under the internal U(1) action identified in the symmetry breaking. The is in contrast to Kaluza-Klein theories in which a 4-vector object A_μ, exhibiting properties of gauge field, can be identified in the extra components of a higher-dimensional metric tensor, as for example in the 5×5 metric case as originally formulated in the 1920s [1, 2]. In the following section we consider a further natural extension of equation 19 for a higher-dimensional form of $L(\boldsymbol{v}) = 1$ for the present theory.

4 E_6 Symmetry on $\mathrm{h}_3\mathbb{O}$

A natural generalisation from the space $\mathrm{h}_3\mathbb{C}$, underlying the vector $\boldsymbol{v}_9 \in \mathrm{h}_3\mathbb{C}$ transformed in equation 19 as a symmetry time, is obtained by augmenting the complex numbers $\mathbb{C}$ to the largest division algebra, namely the octonions $\mathbb{O}$ [3]. The vector space $\mathrm{h}_3\mathbb{O}$ obtained corresponds to the set of 3×3 Hermitian matrices over the octonions with elements which can be written as (in this paper we closely follow [4], [5] chapters 3 and 4, together with [6, 7, 8], for all details of the E_6 structure, and generally adopt the notation therein):

$$\mathcal{X} \;=\; \begin{pmatrix} p & \bar{a} & c \\ a & m & \bar{b} \\ \bar{c} & b & n \end{pmatrix} \;=\; \left(\begin{array}{c|c} X & \theta \\ \hline \theta^\dagger & n \end{array}\right) \in \mathrm{h}_3\mathbb{O} \tag{20}$$

with $p, m, n \in \mathbb{R}$ (here the component labels are chosen to conform with the notation in the main references, and n here is of course not the dimension of any space), $a, b, c \in \mathbb{O}$ and $\bar{a}$ denotes the octonion conjugate of a reversing the sign of the 7-dimensional imaginary part. In general an octonion can be described by eight real parameters $\{a_1 \ldots a_8\}$ and written as:

$$a \;=\; a_1 \;+\; a_2\, i \;+\; a_3\, j + \; a_4\, k \;+\; a_5\, kl \;+\; a_6\, jl \;+\; a_7\, il \;+\; a_8\, l \tag{21}$$

The seven imaginary units in this basis $\{i, j, k, kl, jl, il, l\}$, with $i^2 = j^2 = \ldots = il^2 = l^2 = -1$, are mutually anticommuting, with $il\, j = -j\, il$ etc., with their full algebraic composition described in [5, 6]. While the octonions are non-associative they compose the largest 'normed division algebra' [3]. In equation 20 X and θ have the structure of an octonionic 2×2 vector and 1×2 spinor respectively. Hence the vector space $\mathrm{h}_3\mathbb{O}$ is 27-dimensional over the real numbers. It is a space with particularly rich symmetry properties largely owing to the nature of the 8-dimensional octonion subspaces [3].

As for the space $\mathrm{h}_3\mathbb{C}$ a cubic norm, or determinant $\det(\mathcal{X})$ for $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$, can be defined on the space $\mathrm{h}_3\mathbb{O}$ taking the form:

$$\begin{aligned} \det(\mathcal{X}) &= \det(X)n + 2X \cdot (\theta\theta^\dagger) && (22) \\ &= pmn - p|b|^2 - m|c|^2 - n|a|^2 + 2\mathrm{Re}(\bar{a}\bar{b}\bar{c}) && (23) \end{aligned}$$

where the 10-dimensional Lorentz inner product $X \cdot Y$ with $X, Y \in \mathrm{h}_2\mathbb{O}$, in the first line, together with equation 20 can be used to derive the second line in which the cubic composition of components, consistent with the homogeneous form of equation 11 (and with equation 2 as a particular cubic form), is explicitly seen. The symmetry of the form $L(\boldsymbol{v}_{27}) = 1$, that is the symmetry leaving $\det(\mathcal{X})$ invariant, is a real form of E_6 as we briefly review here. We closely follow references [4, 5, 6, 7, 8] within which, in particular, the means of accommodating, and employing, the non-associative property of the octonions is described in detail.

As a generalisation from the $\mathrm{SL}(2, \mathbb{C}) \equiv \mathrm{Spin}^+(1, 3)$ Lorentz transformations of equation 19 a set of 2×2 matrix actions with $M \in \mathrm{SL}(2, \mathbb{O}) \equiv \mathrm{Spin}^+(1, 9)$ can be embedded in the upper-left corner of 3×3 matrices $\mathcal{M}$ to obtain a conjugation action for the 3×3 case $R : \mathcal{X} \to \mathcal{M}\mathcal{X}\mathcal{M}^\dagger$ with:

$$\mathcal{M}\mathcal{X}\mathcal{M}^\dagger = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} X & \theta \\ \hline \theta^\dagger & n \end{array}\right) \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right)^\dagger = \left(\begin{array}{c|c} MXM^\dagger & M\theta \\ \hline \theta^\dagger M^\dagger & n \end{array}\right) \tag{24}$$

This expression contains the vector $X \to R(X) = MXM^\dagger$, spinor $\theta \to R(\theta) = M\theta$ and scalar $n \to 1n$ representations of $\mathrm{SL}(2, \mathbb{O})$, each transforming in the appropriate way with the form of the action R determined correspondingly. These transformations respect the vector and spinor block structure described in equation 20.

With three natural ways to embed the transformations $\mathrm{SL}(2, \mathbb{O}) \subset \mathrm{SL}(3, \mathbb{O})$ the action depicted in equation 24 is denoted as 'type 1' The form of $\mathrm{h}_3\mathbb{O}$ matrices transforming under the type 1 $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{O})$ actions is compatible with the

isomorphism of vector spaces ([3] p.30):

$$\mathrm{h}_3\mathbb{O} \quad \cong \quad \mathbb{R} \oplus \mathrm{h}_2\mathbb{O} \oplus \mathbb{O}^2 \tag{25}$$

$$\begin{pmatrix} \begin{pmatrix} & X & \end{pmatrix} & \begin{pmatrix} \theta \end{pmatrix} \\ \begin{pmatrix} & \theta^\dagger & \end{pmatrix} & n \end{pmatrix} \quad \rightarrow \quad (n, \quad X, \quad \theta) \tag{26}$$

$$\mathbf{27}_{\mathrm{E}_6} \quad \rightarrow \quad (\mathbf{1} + \mathbf{10} + \mathbf{16})_{\mathrm{Spin}^+(1,9)} \tag{27}$$

The three parts of this decomposition are respectively the scalar, vector and spinor representations of the 10-dimensional spacetime symmetry group $\mathrm{SO}^+(1,9)$, for which the covering group is $\mathrm{Spin}^+(1,9) \equiv \mathrm{SL}(2,\mathbb{O})$. The object $\theta = \binom{c}{\bar{b}}$, from equation 20, corresponds to the Majorana-Weyl spinor representation, also denoted as **16**.

With the group $\mathrm{SL}(2,\mathbb{O})$ being 45-dimensional and with three distinct types of embedding this implies to a total of $45 \times 3 = 135$ determinant preserving actions on the space $\mathrm{h}_3\mathbb{O}$. The collective symmetry of these actions is described in terms of vector fields on the tangent space to $\mathrm{h}_3\mathbb{O}$:

$$\dot{R} = \frac{\partial \left(R(\alpha)\mathcal{X}\right)}{\partial \alpha}\bigg|_{\alpha=0} \in T\mathrm{h}_3\mathbb{O} \tag{28}$$

where $\alpha \in \mathbb{R}$ parametrises a given action. By studying the linear dependences of the resulting 135 vector fields it can be shown [5, 6, 7, 8] how a linearly independent basis of 78 actions emerges which fully describes $\mathrm{E}_6 \equiv \mathrm{SL}(3,\mathbb{O})$ as the determinant preserving transformations on $\mathrm{h}_3\mathbb{O}$. The entire group is then described in terms of the actions of matrices $\mathcal{M}$ on the space $\mathrm{h}_3\mathbb{O}$, with the preferred basis for the Lie algebra represented on $T\mathrm{h}_3\mathbb{O}$ reproduced below in table 1.

Here all 78 generators are explicitly determined and listed in tables 6 and 7 in the appendix of this paper, for the category $\{1,2\}$ and 3 transformations respectively, as tangent vector fields $\dot{R} \in T\mathrm{h}_3\mathbb{O}$ which, from equation 20, are of the form:

$$\dot{R} = \begin{pmatrix} \dot{p} & \dot{\bar{a}} & \dot{c} \\ \dot{a} & \dot{m} & \dot{\bar{b}} \\ \dot{\bar{c}} & \dot{b} & \dot{n} \end{pmatrix} \in T\mathrm{h}_3\mathbb{O} \tag{29}$$

The Lie algebra commutator, which determines the structure constants of the E_6 Lie algebra, for any two elements $\dot{R}_1, \dot{R}_2$ is defined through the action of the respective one-parameter subgroups $R_1(\alpha)$ and $R_2(\alpha)$ at any point $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$:

$$[\dot{R}_2, \dot{R}_1] \quad = \quad \frac{\partial}{\partial(\alpha^2)}[R_2(-\alpha) \circ R_1(-\alpha) \circ R_2(\alpha) \circ R_1(\alpha)\, \mathcal{X}]\Big|_{\alpha=0} \tag{30}$$

The full E_6 Lie algebra commutation table is available in [5], for which the full set of $(78 \times 78 - 78)/2 = 3003$ independent entries were found by computer program.

			#
Category 1: Boosts			
$\dot{B}^1_{\underline{tz}}$	$\dot{B}^1_{\underline{tx}}$	$\dot{B}^1_{\underline{tq}}$	9
$\dot{B}^2_{\underline{tz}}$	$\dot{B}^2_{\underline{tx}}$	$\dot{B}^2_{\underline{tq}}$	9
	$\dot{B}^3_{\underline{tx}}$	$\dot{B}^3_{\underline{tq}}$	8
Category 2: Rotations			
$\dot{R}^1_{\underline{xq}}$	$\dot{R}^1_{\underline{xz}}$	$\dot{R}^1_{\underline{zq}}$	15
	$\dot{R}^2_{\underline{xz}}$	$\dot{R}^2_{\underline{zq}}$	8
	$\dot{R}^3_{\underline{xz}}$	$\dot{R}^3_{\underline{zq}}$	8
Category 3: Transverse Rotations			
$\dot{A}_q$	$\dot{G}_q$	$\dot{S}^1_q$	21
		Total Generators	78

Table 1: The complete basis for the Lie algebra of E_6, in terms of tangent vector fields on $T\mathrm{h}_3\mathbb{O}$, reproduced from ([5] p.177, table A.1). The superscripts denote the 'type'. The subscript q denotes any of the seven imaginary octonion units $\{i, j, k, \underline{kl}, \underline{jl}, \underline{il}, l\}$.

For this paper sign and other conventions have been tuned both for internal consistency and for consistency with the entries of the full E_6 algebra table [5]. (In this paper 'E_6' may refer to either the Lie group or the Lie algebra, depending on the context, with a similar convention for the other exceptional Lie groups).

5 E_6 Symmetry Breaking Structure

We can identify the Lorentz 4-vector $\boldsymbol{v}_4 = (v^0, v^1, v^2, v^3) \equiv \boldsymbol{h}_2$ in the upper left-hand 2×2 matrix embedded within the larger 3×3 matrices in $\mathrm{h}_3\mathbb{O}$, as was the case for $\mathrm{h}_2\mathbb{C} \subset \mathrm{h}_3\mathbb{C}$ in equation 19. However here the preferred subspace $\mathbb{C} \subset \mathbb{O}$ basis is taken to be $\{1, l\}$ for $\boldsymbol{v}_4$, with l one of the imaginary octonion units introduced in equation 21, as indicated in equation 32. The relation $\det(\mathcal{X}) = 1$ with $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ is preserved under operations of $\mathrm{SL}(2, \mathbb{C})$ representing the Lorentz group upon this space as:

$$\mathcal{X} \rightarrow \left(\begin{array}{c|c} S & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} h^{00} & h^{01} + \overline{a}(6) & c \\ h^{10} + a(6) & h^{11} & \overline{b} \\ \hline \overline{c} & b & n \end{array}\right) \left(\begin{array}{c|c} S^\dagger & 0 \\ \hline 0 & 1 \end{array}\right) \tag{31}$$

$$\text{with} \qquad \begin{array}{ll} h^{00} = v^0 + v^3, & h^{01} = v^1 - v^2 l \\ h^{10} = v^1 + v^2 l, & h^{11} = v^0 - v^3 \end{array} \tag{32}$$

with $S \in \mathrm{SL}(2,\mathbb{C})$, and with '1' describing the identity transformation in the trivial 1-dimensional representation of this group, acting upon the components of $\mathcal{X}$ of equation 20. This action preserves the value of $\det(\boldsymbol{h}_2) = h^2$, as it is simply the transformation of equation 18, as well as leaving $\det(\mathcal{X}) = 1$ invariant. In equation 31 $a(6)$ denotes the 6-dimensional imaginary part of $a \in \mathrm{h}_3\mathbb{O}$ of equation 20, that is excluding the real $a_1 = v^1$ and imaginary $a_8 l = v^2 l$ components of $a \in \mathbb{O}$ which are associated with the external 4-vector $\boldsymbol{v}_4 \in TM_4$. Since the $\mathrm{SL}(2,\mathbb{C})$ actions, based on the $\{1, l\}$ complex subspace, are embedded in the 'type 1' location this group will be denoted $\mathrm{SL}(2,\mathbb{C})^1$. This describes the Lorentz subgroup for 4-dimensional spacetime generated by the subset of Lie algebra elements in E_6 from table 1:

$$\{\dot{B}^1_{\underline{tz}}, \dot{R}^1_{\underline{xl}}, \dot{B}^1_{\underline{tx}}, \dot{B}^1_{\underline{tl}}, \dot{R}^1_{\underline{xz}}, \dot{R}^1_{\underline{zl}}\} \tag{33}$$

With the group action of $\mathrm{SL}(2,\mathbb{C})^1$ on $\mathrm{h}_3\mathbb{O}$ in equation 31 embedded within the type 1 group action of $\mathrm{SL}(2,\mathbb{O})^1$ on the same space as displayed in equation 24 we can write:

$$\mathrm{SO}^+(1,3) \equiv \mathrm{SL}(2,\mathbb{C})^1 \subset \mathrm{SL}(2,\mathbb{O})^1 \subset \mathrm{SL}(3,\mathbb{O}) \equiv \mathrm{E}_6 \tag{34}$$

where the first '$\equiv$' strictly applies at the Lie algebra level. This shows explicitly how the action of the Lorentz group may be embedded within the higher symmetry group E_6 acting on the space $\mathrm{h}_3\mathbb{O}$. The direct physical interpretation of the former symmetry as an action on the external spacetime manifold M_4 (as originally depicted in figure 1(b)) provides a direct source for the breakdown of the latter symmetry.

The two-sided $\mathrm{SL}(2,\mathbb{C})^1$ action on $\mathrm{h}_2\mathbb{O}$ in equation 31 only transforms the real diagonal entries h^{00} and h^{11} together with the $h^{10} = a_1 + a_8 l$ and $h^{01} = a_1 - a_8 l$ components of $a \in \mathbb{O}$. The six components of $a(6) \in \mathrm{Im}(a)$ remain invariant as may be deduced from the form of the six $\mathrm{SL}(2,\mathbb{C})^1$ generators in table 6. (This is equivalent to the invariance of $\underline{\boldsymbol{v}}_6$ under $\mathrm{SO}^+(1,3)$ for the model of figure 1). Of the additional 17 components in $\mathrm{h}_3\mathbb{O}$ the real diagonal n entry is also invariant, as is clear from equation 31, while *all* 16 components of $b, c \in \mathbb{O}$ transform non-trivially under the one-sided $\mathrm{SL}(2,\mathbb{C})^1$ action.

The spinor $\theta_l = \binom{c}{\bar{b}}_l \in \mathbb{C}^2$ will denote the $\{1, l\}$ components of c and $\bar{b}$ in θ, that is $\binom{c}{\bar{b}} \in \mathbb{O}^2$ restricted to the $\{1, l\}$ complex subspace. By comparison with equation 19 this object transforms as a left-handed Weyl spinor $\psi_L = \theta_l$ under the $\mathrm{SL}(2,\mathbb{C})^1$ action in equation 31. Here we take this $S \in \mathrm{SL}(2,\mathbb{C})^1$ action on θ_l to *define* the left-handed spinor representation.

Considering the three quaternionic subspaces with the base units $\{1, l, i, \underline{il}\}$, $\{1, l, j, \underline{jl}\}$ and $\{1, l, k, \underline{kl}\}$ it can be seen through explicit calculation that the original full octonionic spinor $\theta = \binom{c}{\bar{b}}$, with 16 real components, reduces to a total of four left-handed Weyl spinors under the action of $\mathrm{SL}(2,\mathbb{C})^1$, namely:

$$\theta_l = \begin{pmatrix} c_1 + c_8 l \\ b_1 - b_8 l \end{pmatrix}, \quad \theta_i = \begin{pmatrix} c_7 \underline{il} + c_2 i \\ -b_7 \underline{il} - b_2 i \end{pmatrix}, \quad \theta_j = \begin{pmatrix} c_6 \underline{jl} + c_3 j \\ -b_6 \underline{jl} - b_3 j \end{pmatrix}, \quad \theta_k = \begin{pmatrix} c_5 \underline{kl} + c_4 k \\ -b_5 \underline{kl} - b_4 k \end{pmatrix} \tag{35}$$

with the respective components of each transforming in an identical manner. Overall the decomposition of the **27** representation of E_6 under the subgroup $\mathrm{Spin}^+(1,9)$ of equation 27 further reduces under the subgroup $\mathrm{SL}(2,\mathbb{C})^1$ as summarised in table 2.

$\mathrm{Spin}^+(1,9)$	$\mathrm{SL}(2,\mathbb{C})^1$	Components
1 scalar	(0,0) scalar	n
10 vector	$(\frac{1}{2},\frac{1}{2})$ vector	$\boldsymbol{v}_4$
	$6\times(0,0)$ scalars	$a(6)$
16 spinor	$4\times(\frac{1}{2},0)$ spinors	$\theta_{l,i,j,k}$

Table 2: The further decomposition of the $(\mathbf{1}+\mathbf{10}+\mathbf{16})$ representation of $\mathrm{Spin}^+(1,9)\subset \mathrm{E}_6$ of equation 27 under the subgroup of external 4-dimensional spacetime symmetry $\mathrm{SL}(2,\mathbb{C})^1\subset \mathrm{Spin}^+(1,9)$ actions of equation 31, and the corresponding components of $\mathrm{h}_3\mathbb{O}$ transformed.

As a preliminary definition, and in contrast to the external symmetry, the internal symmetry will be obtained from the set of all E_6 actions on $\mathrm{h}_3\mathbb{O}$ which leave the four components for any $\boldsymbol{v}_4=(v^0,v^1,v^2,v^3)$ in equation 32 (that is, $\boldsymbol{h}_2\in \mathrm{h}_2\mathbb{C}$ of equation 31) invariant. These components, including $v^2\equiv a_8$ associated with the imaginary unit l of $a\in\mathbb{O}$, can also be expressed in the combination (p,m,a_1,a_8) with respect to the parametrisation of equations 20 and 21. The corresponding symmetry group is complementary to the actions of $\mathrm{SL}(2,\mathbb{C})^1$ and will be denoted $\mathrm{Stab}(TM_4)$ as the stability group of all vectors $\boldsymbol{v}_4\in TM_4$. By inspection from tables 6 and 7, for the 78 elements in the preferred basis for the Lie algebra of E_6 defined on the space $T\mathrm{h}_3\mathbb{O}$, the group $\mathrm{Stab}(TM_4)$ is generated by the 31 elements listed in table 3.

Category 1 and 2: Boosts and Rotations	#
$(\dot{R}^2_{\underline{xz}}-\dot{B}^2_{\underline{tx}})$, $(\dot{R}^3_{\underline{xz}}+\dot{B}^3_{\underline{tx}})$	2
$(\dot{R}^2_{\underline{zq}}+\dot{B}^2_{\underline{tq}})$, $(\dot{R}^3_{\underline{zq}}-\dot{B}^3_{\underline{tq}})$	14
Category 3: Transverse Rotations	
$\dot{A}_q$, $\dot{G}_l$, $\dot{S}^1_l$	9
$(\dot{G}_q+2\dot{S}^1_q)$ $q=\{i,j,k,\underline{kl},\underline{jl},\underline{il}\}$	6
Total	31

Table 3: The Lie algebra generators of the 31 dimensional group $\mathrm{Stab}(TM_4)$. The subscript q denotes any of the seven imaginary octonion units $\{i,j,k,\underline{kl},\underline{jl},\underline{il},l\}$ unless stated otherwise.

Of particular interest is the SU(3) subgroup associated with the 8 generators $\{\dot{A}_q,\dot{G}_l\}$ (as also described in [5] pp.115 and 136, following [9] in relation to the SU(3) colour symmetry). This SU(3) is defined in terms of the transverse rotations acting on the octonion space $\mathbb{O}$ alone as the subgroup $\mathrm{SU}(3)\subset \mathrm{G}_2$ of the octonion automorphism group that leaves one imaginary unit, here l, invariant. The corresponding Lie algebra su(3), described by the set of 8 generators $\{\dot{A}_q,\dot{G}_l\}$, as transformations of E_6 on the

full space $h_3\mathbb{O}$, act on each of the octonion elements $a, b, c \in \mathbb{O}$ in the same way (as described in the table 7 caption) leaving invariant the complex $\{1, l\}$ subspaces, and as elements of table 3 identified within $\mathrm{stab}(TM_4)$ may be provisionally associated with the colour $\mathrm{su}(3)_c$ of the Standard Model for the present theory. This algebra is also independent of $\mathrm{SL}(2,\mathbb{C})^1$ in terms of the Lie bracket composition, that is $[X, Y] = 0$ for all $X \in \mathrm{sl}(2,\mathbb{C})^1$ and $Y \in \mathrm{su}(3)_c$, and hence we have the semi-simple subgroup:

$$\mathrm{SL}(2,\mathbb{C})^1 \times \mathrm{SU}(3)_c \subset \mathrm{E}_6 \tag{36}$$

The $\{\dot{A}_q, \dot{G}_l\}$ algebra elements, as transformations on the space $h_3\mathbb{O}$ mix the components of the $\mathrm{Spin}^+(1,9)$ spinor $\theta = \binom{c}{\bar{b}} \in \mathbb{O}^2$. For example the tangent vector field $\dot{A}_i$ on the $\binom{c}{\bar{b}}$ components of $h_3\mathbb{O}$, obtained from table 7, are:

$$\begin{aligned}
\dot{A}_i : \begin{pmatrix} \dot{c} \\ \dot{\bar{b}} \end{pmatrix} &= \begin{pmatrix} \dot{c}_1 + \dot{c}_8 l, & +\dot{c}_7 il + \dot{c}_2 i, & +\dot{c}_6 jl + \dot{c}_3 j, & +\dot{c}_5 kl + \dot{c}_4 k \\ \dot{b}_1 - \dot{b}_8 l, & -\dot{b}_7 il - \dot{b}_2 i, & -\dot{b}_6 jl - \dot{b}_3 j, & -\dot{b}_5 kl - \dot{b}_4 k \end{pmatrix} \\
&= \begin{pmatrix} 0 + 0l, & +0il + 0i, & -c_5 jl - c_4 j, & +c_6 kl + c_3 k \\ 0 - 0l, & -0il - 0i, & +b_5 jl + b_4 j, & -b_6 kl - b_3 k \end{pmatrix}
\end{aligned} \tag{37}$$

The components here have been ordered to match those of the four left-handed Weyl spinors $(\theta_l, \theta_i, \theta_j, \theta_k)$ of equation 35. The fact that each real component of c transforms in the same way as the corresponding component of b is expected since $\mathrm{SU}(3)_c$ acts on each of $a, b, c \in \mathbb{O}$ in precisely the same way. However, it is also noted that the action $\dot{A}_i$ in equation 37 respects the 4-way spinor decomposition.

The extraction of the components of a spinor θ into a matrix of real numbers will be denoted by $[\theta]$. For example, from equation 35 the spinor θ_i can be mapped to the 2×2 matrix of real numbers $[\theta_i] = \left(\begin{smallmatrix} c_7 & c_2 \\ -b_7 & -b_2 \end{smallmatrix}\right)$ (with components ordered to match those of the spinor θ_l under $\mathrm{SL}(2,\mathbb{C})^1$ transformations, as described for equation 35). Using this notation, and with 0_2 representing the 2×2 zero matrix, the tangent vectors of all eight generators $\{\dot{A}_q, \dot{G}_l\}$ of $\mathrm{SU}(3)_c$, including $\dot{A}_i$ from equation 37, on the spinor space $\theta \in \mathbb{O}^2$ are listed in table 4 alongside the actions of the Gell-Mann matrices λ_α, using the correspondence of ([5] p.137, table 4.5), on the vectors $\boldsymbol{u} \in \mathbb{C}^3$. On the left-hand side the elements $\{\dot{A}_q, \dot{G}_l\}$ are already expressed as tangent vectors, while on the right-hand side the tangents are obtained by matrix multiplication of the λ_α into $\boldsymbol{u} \in \mathbb{C}^3$.

In table 4 a term such as $[l\theta_i]$ denotes multiplying the spinor θ_i on the left by l before extracting the coefficients of $l\theta_i$ with the $\mathrm{Im}(\mathbb{O})$ units ordered as in equation 35. This notation is used to isolate the mixing effect on the real number coefficients, with care for the joint effects of the division algebra composition as well as matrix algebra composition. The internal $\mathrm{SU}(3)_c$ symmetry action on the left-hand side of table 4 dovetails neatly with the external $\mathrm{SL}(2,\mathbb{C})^1$ spinor structure of equation 35. The mixing action of $\mathrm{SU}(3)_c$ in table 4 takes a form summarised as:

$$\theta = (\theta_l, \ \underbrace{\theta_i, \ \ \theta_j, \ \ \theta_k}_{\mathrm{SU}(3)_c \text{ action}}) \tag{38}$$

	$([\dot{\theta}_l],$	$[\dot{\theta}_i],$	$[\dot{\theta}_j],$	$[\dot{\theta}_k])$			$(\ \dot{u}_1,$	$\dot{u}_3,$	$\dot{u}_3\ )$
$\dot{A}_i =$	$(\ 0_2,$	$0_2,$	$-[\theta_k],$	$[\theta_j])$	$\sim$	$\lambda_4 \Rightarrow$	$(\ u_3,$	$0,$	$u_1\ )$
$\dot{A}_{\underline{il}} =$	$(\ 0_2,$	$0_2,$	$[l\theta_k],$	$[l\theta_j])$	$\sim$	$\lambda_5 \Rightarrow$	$(-iu_3,$	$0,$	$iu_1\ )$
$\dot{A}_j =$	$(\ 0_2,$	$[\theta_k],$	$0_2,$	$-[\theta_i])$	$\sim$	$\lambda_7 \Rightarrow$	$(\ 0,$	$-iu_3,$	$iu_2\ )$
$\dot{A}_{\underline{jl}} =$	$(\ 0_2,$	$-[l\theta_k],$	$0_2,$	$-[l\theta_i])$	$\sim$	$\lambda_6 \Rightarrow$	$(\ 0,$	$u_3,$	$u_2\ )$
$\dot{A}_k =$	$(\ 0_2,$	$-[\theta_j],$	$[\theta_i],$	$0_2)$	$\sim$	$\lambda_1 \Rightarrow$	$(\ u_2,$	$u_1,$	$0\ )$
$\dot{A}_{\underline{kl}} =$	$(\ 0_2,$	$[l\theta_j],$	$[l\theta_i],$	$0_2)$	$\sim$	$\lambda_2 \Rightarrow$	$(-iu_2,$	$iu_1,$	$0\ )$
$\dot{A}_l =$	$(\ 0_2,$	$[l\theta_i],$	$-[l\theta_j],$	$0_2)$	$\sim$	$\lambda_3 \Rightarrow$	$(\ u_1,$	$-u_2,$	$0\ )$
$\dot{G}_l =$	$(\ 0_2,$	$[l\theta_i],$	$[l\theta_j],$	$-2[l\theta_k])$	$\sim$	$\lambda_8 \Rightarrow$	$\frac{1}{\sqrt{3}}(\ u_1,$	$u_2,$	$-2u_3)$

Table 4: The tangent vector generators for the $SU(3)_c$ representations on $\mathbb{O}^2$ and $\mathbb{C}^3$. The column vectors of $\mathbb{C}^3$ are displayed as a row vectors for convenience in the table.

which implies that as a gauge theory the $SU(3)_c$ internal symmetry will mediate interactions between the Weyl spinors $\theta_i, \theta_j, \theta_k$, transforming under the fundamental representation, which in turn will hence be identified with the three colour degrees of freedom of the quark states. On the other hand the invariance of θ_l, transforming under the trivial representation of $SU(3)_c$, suggests that these components should be associated with the leptonic sector of the Standard Model (with the subscript l originating from the $\{1, l\}$ base units for θ_l also then serving as a mnemonic for its leptonic character).

Of the 31 generators for the internal symmetry group $\mathrm{Stab}(TM_4)$ listed in table 3 there is a $(31-8) = 23$-dimensional set which as a vector space is independent of the internal $SU(3)_c$ generators. However, of these 23 elements $\dot{S}_l^1$ as the *only* E_6 Lie algebra generator of $\mathrm{Stab}(TM_4)$ which is independent of both $SL(2,\mathbb{C})^1$ and $SU(3)_c$. Hence of the many possible $U(1) \subset E_6$ subgroups the internal $U(1)$ generated by $\dot{S}_l^1$ is a natural candidate to consider for the $U(1)_Q$ component of the Standard Model gauge symmetry group associated with electromagnetism. From table 7 it can be seen that the generator $\dot{S}_l^1$ impacts on all 8 real components of both c and $\bar{b}$ of $\theta \in \mathbb{O}^2$. In fact, and in comparison with equation 37, the tangent vector $\dot{S}_l^1$ on the spinor components

$\theta = \binom{c}{\bar{b}}$ is given explicitly by:

$$
\begin{aligned}
\dot{S}_l^1 : \begin{pmatrix} \dot{c} \\ \dot{\bar{b}} \end{pmatrix} &= \begin{pmatrix} \dot{c}_1 + \dot{c}_8 l, & +\dot{c}_7 il + \dot{c}_2 i, & +\dot{c}_6 jl + \dot{c}_3 j, & +\dot{c}_5 kl + \dot{c}_4 k \\ \dot{b}_1 - \dot{b}_8 l, & -\dot{b}_7 il - \dot{b}_2 i, & -\dot{b}_6 jl - \dot{b}_3 j, & -\dot{b}_5 kl - \dot{b}_4 k \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{2}c_8 + \frac{3}{2}c_1 l, & +\frac{1}{2}c_2 il - \frac{1}{2}c_7 i, & +\frac{1}{2}c_3 jl - \frac{1}{2}c_6 j, & +\frac{1}{2}c_4 kl - \frac{1}{2}c_5 k \\ \frac{3}{2}b_8 + \frac{3}{2}b_1 l, & -\frac{1}{2}b_2 il + \frac{1}{2}b_7 i, & -\frac{1}{2}b_3 jl + \frac{1}{2}b_6 j, & -\frac{1}{2}b_4 kl + \frac{1}{2}b_5 k \end{pmatrix} \quad (39) \\
\text{with} \quad [\dot{\theta}] &= \Big(\; +\frac{3}{2}\,[l\theta_l], \quad -\frac{1}{2}\,[l\theta_i], \quad -\frac{1}{2}\,[l\theta_j], \quad -\frac{1}{2}\,[l\theta_k] \; \Big) \quad (40)
\end{aligned}
$$

which may be compared with the $\mathrm{su}(3)_c$ action on θ in equation 37 and table 4. Here the two components of $c \in \mathbb{O}$ *within* each of the four Weyl spinors are mixed, and similarly for the corresponding pair of $\bar{b} \in \mathbb{O}$ components, with no mixing of components *between* different spinors. This is consistent with the nature of the electromagnetic interaction which does not transform between different fermion types.

A further observation from equation 40 regards the factor of $\frac{3}{2}$ found for the θ_l spinor in contrast to the factors of $\frac{1}{2}$ aligned with the three remaining spinors $\theta_i, \theta_j, \theta_k$. Hence, with $\dot{S}_l^1$ provisionally associated with electromagnetism and by comparison with equation 38, the apparent 'electromagnetic charge' assigned to the leptonic sector is *three times* larger than that assigned to the quark sector. Associating $\theta_i, \theta_j, \theta_k$ with the three colour states of a d-quark this observation in principle accounts for the 'fractional charge' of magnitude $\frac{1}{3}$ as theoretically ascribed and empirically confirmed for d-quark states relative to the electron charge. Based on this observation we introduce the notation:

$$\dot{\not{S}}_l^a = \frac{2}{3}\dot{S}_l^a \quad (41)$$

(for $a = 1, 2, 3$) such that the above charge values $\frac{3}{2}$ and $\frac{1}{2}$ are normalised to 1 and $\frac{1}{3}$ under $\dot{\not{S}}_l^1$, representing the generator of $\mathrm{U}(1)_Q$, for ease of comparison with the Standard Model convention for which the electron charge is -1. The 'bar' through $\dot{\not{S}}_l^1$ is a mnemonic symbol for this normalisation of fractional charges relative to the e^- charge.

Hence the subgroup in equation 36 may be augmented to:

$$\mathrm{SL}(2, \mathbb{C})^1 \times \mathrm{SU}(3)_c \times \mathrm{U}(1)_Q \subset \mathrm{E}_6 \quad (42)$$

with the internal group $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ generated by $\{\dot{A}_q, \dot{G}_l, \dot{\not{S}}_l^1\} \in \mathrm{stab}(TM_4)$. The action of this larger internal symmetry on the four $\mathrm{SL}(2, \mathbb{C})^1$ spinors also augments equation 38 as:

$$
\begin{array}{rcccccc}
\theta & = & (\theta_l, & \underbrace{\theta_i, & \theta_j, & \theta_k}) \\
\mathrm{SU}(3)_c & : & \mathbf{1} & & \mathbf{3} & \\
\mathrm{U}(1)_Q & : & +1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}
\end{array} \quad (43)
$$

With the generator $\dot{\not{S}}_l^1$ hence associated with electromagnetic charge it is instructive to consider this action on the full set of $\mathrm{h}_3\mathbb{O}$ components. From table 7 the

diagonal components of $\dot{S}^1_l$ are trivial, with $\dot{p} = \dot{m} = \dot{n} = 0$ in equation 29 for this generator, while the action on the remaining components $a, b, c \in \mathbb{O}$, via equation 41, may be summarised as:

$$\dot{S}^1_l \;=\; \begin{pmatrix} \dot{a} \\ \\ \dot{b} \\ \\ \dot{c} \end{pmatrix} \;=\; \begin{pmatrix} 0\,l\,a_{1,l} & + & \frac{2}{3}\,l\,a(6) \\ \\ -1\,l\,b_{1,l} & - & \frac{1}{3}\,l\,b(6) \\ \\ +1\,l\,c_{1,l} & - & \frac{1}{3}\,l\,c(6) \end{pmatrix} \tag{44}$$

where $a_{1,l} \equiv (a_1 + a_8 l)$ and $a(6) \equiv (a_7 il + a_2 i + a_6 jl + a_3 j + a_5 kl + a_4 k)$, with similar expressions for b and c, following the component order of the spinors in equation 35. The same definition of $a(6)$ is implied in equation 31. By comparison with the above discussion leading to equation 43 the expression for $\dot{S}^1_l$ in equation 44 incorporates 'charges' of 0 and $\frac{2}{3}$ for the $\dot{a}$ components, that is we have:

$$\begin{array}{rcl} a & = & (\,a_{1,l},\; \underbrace{a_{il,i},\; a_{jl,j},\; a_{kl,k}}\,) \\ \mathrm{SU}(3)_c & : & \quad\;\mathbf{1} \qquad\qquad\; \mathbf{3} \\ \mathrm{U}(1)_Q & : & \quad\;0 \quad +\frac{2}{3} \quad +\frac{2}{3} \quad +\frac{2}{3} \end{array} \tag{45}$$

where the $\mathrm{SU}(3)_c$ action on $a \in \mathbb{O}$ is identical to that on the octonion components of $\theta = \binom{c}{b}$ in equation 43. While physical lepton states are invariant under $\mathrm{SU}(3)_c$ and are hence associated with the Weyl spinor θ_l in equation 43, the neutrino states are also invariant under the $\mathrm{U}(1)_Q$ of electromagnetism, that is with zero charge, and are provisionally associated with the $a_{1,l}$ components in equations 44 and 45; while a set of u-quarks with $\frac{2}{3}$ fractional charges is similarly associated with the $a(6)$ components.

However, unlike $\theta = \binom{c}{b}$ the $a \in \mathrm{h}_3\mathbb{O}$ component does *not* correspond to a set of $\mathrm{SL}(2,\mathbb{C})^1$ Weyl spinors, as can be seen from table 2. Further, the 'neutrino' components $a_{1,l} = a_1 + a_8 l = v^1 + v^2 l$ have already apparently been accounted for as part of the external vector $\boldsymbol{v}_4 \in TM_4$ on the base manifold, as described in equations 31 and 32. These features clearly require further investigation.

Within the above caveats, aligned with the charge magnitudes of 1 and $\frac{1}{3}$ for the electron and d-quark Weyl spinors of equation 43 the respective $\mathrm{U}(1)_Q$ charges of 0 and $\frac{2}{3}$ in equation 45 correlate with charges of $\binom{0}{-1}$ for the $\binom{\nu}{e}$ lepton doublet and $\binom{+2/3}{-1/3}$ for the $\binom{u}{d}$ quark doublet of the Standard Model. In addition the states associated with each *left-handed* doublet of charges interact via the exchange of $W^\pm$ gauge bosons in the Standard Model. Hence it remains to be understood how interactions within each of these doublets may be mediated via an $\mathrm{SU}(2)_L$ symmetry, and how such ν-lepton and u-quark components of $a \in \mathbb{O} \subset \mathrm{h}_3\mathbb{O}$ gain a Weyl spinor structure under the external $\mathrm{SL}(2,\mathbb{C})^1$ action. Here we initially focus upon the simple fact that weak $\mathrm{SU}(2)_L$ transformations act on fermion doublets of the form $\binom{\nu}{e}$ and $\binom{u}{d}$, which have been associated with the $\binom{a}{\theta}$ components of $\mathrm{h}_3\mathbb{O}$ in equation 20 for the present theory.

The type 1 $\mathrm{SL}(2,\mathbb{C})^1$ action on the four Weyl spinors of equation 35 is complemented by $\mathrm{SL}(2,\mathbb{C})^2$ and $\mathrm{SL}(2,\mathbb{C})^3$ transformations of type 2 and 3, all involving quaternion algebra composition with $l \in \mathbb{O}$ being the only imaginary octonion unit appearing in the transformation matrices. Two SU(2)s are immediately identifiable

in terms of the rotation subgroups of the type 2 and type 3 Lorentz groups, as denoted by $\mathrm{SU}(2)^2$, generated by the set $\{\dot{R}^2_{zl}, \dot{R}^2_{xz}, \dot{R}^2_{xl}\}$, and $\mathrm{SU}(2)^3$, as generated by $\{\dot{R}^3_{zl}, \dot{R}^3_{xz}, \dot{R}^3_{xl}\}$. Neither $\mathrm{SU}(2)^2 \subset \mathrm{SL}(2,\mathbb{C})^2$ nor $\mathrm{SU}(2)^3 \subset \mathrm{SL}(2,\mathbb{C})^3$ is independent of $\mathrm{SL}(2,\mathbb{C})^1$ within the E_6 Lie algebra, with for example $[\dot{R}^2_{xz}, \dot{R}^1_{xz}] = \frac{1}{2}\dot{R}^3_{xz} \neq 0$, and neither of them forms a subgroup of $\mathrm{Stab}(TM_4)$, and hence they do *not* form an *internal* symmetry by the original definition which led to table 3. However owing to the properties described below in exploring further the structure of these transformations the groups $\mathrm{SU}(2)^{2,3}$ *are* found to be of some interest in relation to the structure of electroweak theory.

The type 1 action of $\mathrm{SL}(2,\mathbb{C})^1$ decomposes the space $\theta^1 = \binom{c}{\bar{b}} \in \mathbb{O}^2$ into the four Weyl spinors of equation 35. The transformations $\mathrm{SL}(2,\mathbb{C})^{2,3}$ of type 2 and 3, with complementary transformation matrices also based on the units $\{1, l\}$, similarly respect the octonion decomposition aligned to the four base unit sets:

$$\{1, l\}, \quad \{il, i\}, \quad \{jl, j\}, \quad \{kl, k\} \tag{46}$$

based on the same quarternion subalgebras, now for all three of $a, b, c \in \mathbb{O}$. Hence the subgroups $\mathrm{SU}(2)^{2,3} \subset \mathrm{E}_6$ describe transformations between the components of equation 43 and those of equation 45 respecting the alignment of the four component pieces, and hence acting independently on the corresponding doublets of leptonic and quark states as appropriate for weak interactions. With respect to the embedding of $a, b, c \in \mathbb{O}$ as components of $\mathrm{h}_3\mathbb{O}$ in equation 20, the spinor representation mixing actions of $\mathrm{SL}(2,\mathbb{C})^{1,2,3}$ can also be displayed graphically as:

$$\left(\begin{tikzcd} & \bar{a} \arrow[r, leftrightarrow, dashed] \arrow[dd, leftrightarrow, dotted] & c \arrow[d, leftrightarrow] \\ a \arrow[rr, leftrightarrow, dotted] \arrow[d, leftrightarrow, dashed] & & \bar{b} \\ \bar{c} \arrow[r, leftrightarrow] & b & \end{tikzcd}\right) \quad \text{with} \quad \begin{tikzcd} {} \arrow[r, leftrightarrow] & \mathrm{SL}(2,\mathbb{C})^1 \\ {} \arrow[r, leftrightarrow, dashed] & \mathrm{SL}(2,\mathbb{C})^2 \\ {} \arrow[r, leftrightarrow, dotted] & \mathrm{SL}(2,\mathbb{C})^3 \end{tikzcd} \tag{47}$$

This indicates how the $\binom{c}{\bar{b}}$ spinor components under $\mathrm{SL}(2,\mathbb{C})^1$ are replaced by $\binom{a}{\bar{c}}$ and $\binom{b}{\bar{a}}$ spinors under $\mathrm{SL}(2,\mathbb{C})^2$ and $\mathrm{SL}(2,\mathbb{C})^3$ respectively, depending on the alignment of the $\theta^a = \binom{\theta_1}{\theta_2}$ components, with θ^1 depicted in equations 20. It is the observation that the $\mathrm{SL}(2,\mathbb{C})^{2,3}$ actions relate the $\theta^1 = \binom{c}{\bar{b}} \in \mathbb{O}^2$ components with the $a \in \mathbb{O}$ component in equations 47, while respecting the four-way octonion decomposition of equation 46, that suggests that these transformations might be closely related to the weak interactions.

The nine generators of the combined type $a = 1, 2$ and 3 rotations $\mathrm{SU}(2)^a \subset \mathrm{SL}(2,\mathbb{C})^a$ form a closed subalgebra of E_6, which is eight dimensional due to the linear dependence of the three $\dot{R}^a_{xl}$ generators. This subalgebra is in fact an su(3), a linearly independent basis for which can be described by the eight rotation generators ([5] p.128):

$$\mathrm{su}(3)_s \equiv \{\dot{R}^1_{xl},\ \dot{R}^2_{xl},\ \dot{R}^1_{xz},\ \dot{R}^2_{xz},\ \dot{R}^3_{xz},\ \dot{R}^1_{zl},\ \dot{R}^2_{zl},\ \dot{R}^3_{zl}\} \tag{48}$$

These generate a group denoted $\mathrm{SU}(3)_s$ (where 's' denotes the 'standard' representation or embedding of SU(3) in E_6 [5]). Since each $\mathrm{SL}(2,\mathbb{C})^a$ is independent of $\mathrm{SU}(3)_c$, within

E_6 the subgroup $\mathrm{SU}(3)_s$ is also independent of the colour subgroup $\mathrm{SU}(3)_c$, as generated by the eight elements on the left-hand side of table 4, with the Lie bracket composition of any element of equation 48 with any element of $\{\dot{A}_q, \dot{G}_l\}$ being zero.

Guided by a similar construction based on $\mathrm{SU}(2)_L$ generators in the Standard Model, here in the complex algebra for the generators of the subgroup $\mathrm{SU}(2)^2 \subset \mathrm{E}_6$ we define:

$$\dot{\Sigma}^{(2)\pm} := \dot{R}^2_{\not{z}l} \;\pm\; i\dot{R}^2_{\not{x}\not{z}} \tag{49}$$

Here the imaginary unit $i \in \mathbb{C}$ in the complexification of the E_6 Lie algebra commutes with the elements of $T\mathrm{h}_3\mathbb{O}$, which are based on an independent octonion algebra $\mathbb{O}$. On reading off the corresponding entries in the Lie algebra table in [5] for the complex element of equation 49 it is found that:

$$\begin{array}{lrcl} & [\dot{S}^1_l \,,\, (\dot{R}^2_{\not{z}l} + i\dot{R}^2_{\not{x}\not{z}})] & = & \frac{3}{2}\dot{R}^2_{\not{x}\not{z}} \;-\; i\frac{3}{2}\dot{R}^2_{\not{z}l} \;=\; -i\frac{3}{2}(\dot{R}^2_{\not{z}l} + i\dot{R}^2_{\not{x}\not{z}}) \qquad (50) \\ \text{hence} & [i\dot{S}^1_l \,,\, (\dot{R}^2_{\not{z}l} + i\dot{R}^2_{\not{x}\not{z}})] & = & +(\dot{R}^2_{\not{z}l} + i\dot{R}^2_{\not{x}\not{z}}) \\ \text{and} & [i\dot{S}^1_l \,,\, \dot{\Sigma}^{(2)\pm}] & = & \pm\dot{\Sigma}^{(2)\pm} \qquad (51) \end{array}$$

with real charge eigenvalues ± 1. Hence the generators $\dot{\Sigma}^{(2)\pm}$ of equation 49 are associated with the *same* magnitude of $\mathrm{U}(1)_Q$ charge under $\dot{S}^1_l$ as was found for the electron in the leptonic components $\theta_l \subset \mathrm{h}_3\mathbb{O}$ as described in equations 39–43. The generators $\dot{\Sigma}^{(2)\pm}$ hence act as charge raising and lowering operators, analogous to the action of $W^\pm$ gauge bosons in the Standard Model, here deriving from the embedding of $\mathrm{SU}(2)^2 \subset \mathrm{SU}(3)_s$.

Further, the E_6 generator linear dependences described in [5] imply the relation:

$$-\dot{S}^1_l = \dot{R}^2_{\not{x}l} + \tfrac{1}{2}\dot{S}^2_l \tag{52}$$

which, within a choice of sign conventions, is closely reminiscent of the Standard Model relation:

$$Q = T^3 + \tfrac{1}{2}Y \tag{53}$$

between the charge Q, the third $\mathrm{SU}(2)_L$ generator T^3 and hypercharge Y. This suggests associating $\dot{S}^2_l$ with Y as a candidate for the generator of the *hypercharge* symmetry $\mathrm{U}(1)^2 \sim \mathrm{U}(1)_Y$ which commutes with $\mathrm{SU}(2)^2$, as generated by $\{\dot{R}^2_{\not{z}l}, \dot{R}^2_{\not{x}\not{z}}, \dot{R}^2_{\not{x}l}\}$.

The $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ symmetry generated by $\{\dot{R}^2_{\not{z}l}, \dot{R}^2_{\not{x}\not{z}}, \dot{R}^2_{\not{x}l}, \dot{S}^2_l\}$ acts on the doublet components of the type 2 spinor $\theta^2 = \binom{a}{\bar{c}}$ in $\mathrm{h}_3\mathbb{O}$. Restricted to the complex subspace $\mathbb{C} \subset \mathbb{O}$ with $\{1, l\}$ basis units the components $\theta^2_l = \binom{a}{\bar{c}}_l$ provisionally represents the lepton doublet $\binom{\nu}{e}$. Since these components do not correspond to complete $\mathrm{SL}(2,\mathbb{C})^1$ Weyl spinors for either the neutrino *or* the electron part this $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ symmetry is clearly not directly equivalent to the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry of electroweak theory. However the components of θ^2_l do transform under the internal symmetry $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ appropriately to represent such a lepton doublet, as described earlier in this section, and hence the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ symmetry serves as a useful intermediate model, considered as a 'mock electroweak theory'.

It is also the case that neither the $\mathrm{SU}(2)^2$ generated by $\{\dot{R}^2_{\not{z}l}, \dot{R}^2_{\not{x}\not{z}}, \dot{R}^2_{\not{x}l}\}$ nor the $\mathrm{U}(1)^2$ generated by $\dot{S}^2_l$ are internal symmetries in the sense of table 3, that is within $\mathrm{Stab}(TM_4)$, with each of these four generators impacting upon the components of the type 1 subspace $\mathrm{h}_2\mathbb{C} \subset \mathrm{h}_3\mathbb{O}$, which represent components of the external spacetime

TM_4, as can be seen explicitly from the form of these four generators in tables 6 and 7. The breaking of the full E_6 symmetry action on $\mathrm{h}_3\mathbb{O}$ in this identification of the type 1 subspace $\mathrm{h}_2\mathbb{C}$ with the local tangent space TM_4 of the external spacetime hence includes the breaking of the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2 \subset \mathrm{E}_6$ subgroup.

With the one generator surviving this $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ symmetry breaking being the $\dot{S}_l^1$ generator on the left-hand side of equation 52, which has been associated with the $\mathrm{U}(1)_Q$ of electromagnetism, this structure is closely analogous to electroweak theory in the Standard Model. This motivates the proposal that the four components of $\boldsymbol{v}_4 \in TM_4 \equiv \mathrm{h}_2\mathbb{C} \subset \mathrm{h}_3\mathbb{O}$ might be considered to form a 'vector-Higgs', in place of the four scalar components of the Standard Model Higgs field, underlying empirically observed Higgs phenomena. The scalar magnitude $|\boldsymbol{v}_4|$, or another scalar deriving from the $\boldsymbol{v}_4$ components might then constitute the Higgs scalar itself, in a manner resembling composite Higgs models. Mass terms for gauge fields associated with the broken $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ generators would then derive from the fact that they do not generate elements of $\mathrm{Stab}(TM_4)$, while fermion masses originate from interaction terms implied in the composition of $L(\boldsymbol{v}_{27}) = 1$ in equation 23, or a higher-dimensional extension.

Using the E_6 Lie algebra table [5] a basis for the $\mathrm{su}(3)_s$ Lie subalgebra can be obtained with a normalised Killing form with components proportional to the unit 8×8 matrix. This can be achieved by replacing the first two basis elements in equation 48 by $\{\dot{R}^a_{\underline{xl}}, \frac{1}{\sqrt{3}}\dot{S}^a_l\}$ for three possible choices corresponding to type $a = 1, 2, 3$. Using this normalisation the value of the mixing angle θ_{M^2} in the breaking of the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ symmetry to $\mathrm{U}(1)_Q$ may be determined, by analogy with the weak mixing angle θ_W in standard electroweak theory.

A significant difference is observed between the calculated value $\sin^2\theta_{M^2} = \frac{3}{4}$ and the known empirical $\sin^2\theta_W \simeq 0.23$. However, as emphasised earlier in this subsection the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ symmetry does *not* act on $\mathrm{SL}(2,\mathbb{C})^1$ Weyl spinors in the appropriate way to describe weak interactions, and here we are dealing with a provisional 'mock electroweak theory', which nevertheless exhibits some of the features associated with corresponding structures of the Standard Model, such as the *possibility* of identifying a mixing angle. It is also noted that in the mock theory the calculated value of $\sin^2\theta_{M^2} = \frac{3}{4}$ effectively corresponds to a 'unification scale' whereas the empirical value of $\sin^2\theta_W \simeq 0.23$ is determined at the practical energy scale of $M_Z \sim 10^2$ GeV.

However while the properties of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$ described above are reminiscent of several features of electroweak group it is well known that in fact the full rank-6 group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, representing the external spacetime symmetry and the internal symmetry of the Standard Model, cannot be accommodated as a subgroup within E_6, for example by the analysis of Dynkin diagrams. This observation, and the need to identify further features of the Standard Model fermions, suggests consideration of a larger symmetry for a higher-dimensional form of time $L(\hat{\boldsymbol{v}}) = 1$ beyond the 27-dimensional cubic form considered in this section.

6 E_7 Symmetry and Left-Right Asymmetry

A higher-dimensional *homogeneous* polynomial form is desired, in conformity with the discussion of equation 2 and more generally with the derivation of equation 11 in section 2. While the determinant preserving symmetry of the space $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ describes the lowest-dimensional non-trivial representation of E_6 the smallest non-trivial representation of the exceptional Lie group E_7 is 56-dimensional and may be constructed in terms the elements x of the Freudenthal triple system $F(\mathrm{h}_3\mathbb{O})$ ([10, 11, 12], [3] p.48).

The vector space $F(\mathrm{h}_3\mathbb{O})$ has 56 real components and may be introduced according to Freudenthal's construction with the vector space composition (which may be compared with the further decomposition of equation 25):

$$F(\mathrm{h}_3\mathbb{O}) \;\cong\; \mathrm{h}_3\mathbb{O} \oplus \mathrm{h}_3\mathbb{O} \oplus \mathbb{R} \oplus \mathbb{R} \tag{54}$$

Correspondingly elements $x \in F$, with $F = F(\mathrm{h}_3\mathbb{O})$, can generally be written in the form of a '2×2 matrix' as:

$$x = \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix}, \qquad \text{with } \mathcal{X}, \mathcal{Y} \in \mathrm{h}_3\mathbb{O}, \quad \alpha, \beta \in \mathbb{R} \tag{55}$$

$$\text{and} \qquad \mathcal{X} = \begin{pmatrix} p & \bar{a} & c \\ a & m & \bar{b} \\ \bar{c} & b & n \end{pmatrix}, \qquad \mathcal{Y} = \begin{pmatrix} P & \bar{A} & C \\ A & M & \bar{B} \\ \bar{C} & B & N \end{pmatrix} \tag{56}$$

here with the real P, M, N and octonion A, B, C components of $\mathcal{Y}$ distinguished from the lower case counterpart components of $\mathcal{X}$. A homogeneous quartic norm $q : F \to \mathbb{R}$ can be defined on the components of $x \in F$ as follows:

$$q(x) \;=\; -2[\alpha\beta - (\mathcal{X}, \mathcal{Y})]^2 \;-\; 8[\alpha \det(\mathcal{X}) + \beta \det(\mathcal{Y}) - (\mathcal{X}^\sharp, \mathcal{Y}^\sharp)] \tag{57}$$

where all the necessary definitions contained within this expression are inherited from those for the Jordan algebra $\mathrm{h}_3\mathbb{O}$ as described in [10, 11, 12]. The group $\mathrm{Inv}(F)$ of all invertible transformations σ in F preserving the above quartic norm with $q(\sigma(x)) = q(x)$ is found to be the non-compact real form $\mathrm{E}_{7(-25)}$ of the exceptional Lie group E_7. Hence, in particular, under this symmetry group the invariance of the quartic form $q(x)$, as a homogeneous polynomial, describes a possible 56-dimensional form of temporal flow in the form of equation 11 which may be denoted $L(\boldsymbol{v}_{56}) = 1$. This provides a natural extension from the cubic form $L(\boldsymbol{v}_{27}) = 1$ with E_6 symmetry described in the previous two sections.

Including the actions of the subgroup $\mathrm{E}_6 \subset \mathrm{E}_7$ on the elements $x \to s(x) \in F$ (here s and x conform with the notation in references [10, 11, 12], their meaning is of course very different to that of s and x in equation 2 for example) the transformations of the full symmetry $\mathrm{E}_7 \equiv \mathrm{Inv}(F)$ may be categorised in terms of four sets. With

$s \in \mathrm{E}_6$, $\lambda \in \mathbb{R}$ and $C, D \in \mathrm{h}_3\mathbb{O}$ these are [10, 11, 12]:

$$T(s): \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & s(\mathcal{X}) \\ s^{*-1}(\mathcal{Y}) & \beta \end{pmatrix} \tag{58}$$

$$\lambda: \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \lambda^{-1}\alpha & \lambda^{\frac{1}{3}}\mathcal{X} \\ \lambda^{-\frac{1}{3}}\mathcal{Y} & \lambda\beta \end{pmatrix} \tag{59}$$

$$\phi(C): \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha + (\mathcal{Y}, C) + (\mathcal{X}, C^\sharp) + \beta \det(C) & \mathcal{X} + \beta C \\ \mathcal{Y} + \mathcal{X} \times C + \beta C^\sharp & \beta \end{pmatrix} \tag{60}$$

$$\psi(D): \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \mathcal{X} + \mathcal{Y} \times D + \alpha D^\sharp \\ \mathcal{Y} + \alpha D & \beta + (\mathcal{X}, D) + (\mathcal{Y}, D^\sharp) + \alpha \det(D) \end{pmatrix} \tag{61}$$

The set of actions s^{*-1} in equation 58 is equivalent to the complex conjugate of the representation defined by the set of actions $s \in \mathrm{E}_6$ on $\mathrm{h}_3\mathbb{O}$. Under the subgroup $\mathrm{E}_{6(-26)} \subset \mathrm{E}_{7(-25)}$ the space F decomposes into the reducible representation ([11] equations 9.45 and 9.46):

$$\mathbf{56}_{\mathrm{E}_7} \rightarrow (\mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1})_{\mathrm{E}_6} \tag{62}$$

compatible with the structure of equation 54 (and can be compared with the further reduction under $\mathrm{Spin}^+(1,9)$ in equation 27).

The four sets of group actions of equations 58–61 are described at the Lie algebra level in reference [13]. In particular it is shown that for the first set of equation 58 the E_6 'rotations' are identical on $\mathcal{X}$ and $\mathcal{Y}$ while for the E_6 'boosts' there is a change in sign between the actions on the two $\mathrm{h}_3\mathbb{O}$ subspaces. Having extended beyond the E_6 subalgebra to the full E_7 we next focus on the generators of the 4-dimensional spacetime Lorentz subgroup $\mathrm{SL}(2,\mathbb{C})^1 \subset \mathrm{E}_6 \subset \mathrm{E}_7$ of type 1 as studied in the first part of the previous section. For these Lorentz actions reversing the sign of the boosts is precisely the operation which interchanges between the left-handed and right-handed spinor representations of $\mathrm{SL}(2,\mathbb{C})$.

Hence while the components of θ_l in θ^1 within $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$, defined in equation 35, transform as a *left*-handed Weyl spinor under $\mathrm{SL}(2,\mathbb{C})^1$ the corresponding components of $\theta_{\mathcal{L}} = \binom{C_1 + C_8 l}{B_1 - B_8 l}$ within the θ^1 component of $\mathcal{Y} \in \mathrm{h}_3\mathbb{O}$, extracted from equation 56, transform as a *right*-handed Weyl spinor under the same $\mathrm{SL}(2,\mathbb{C})^1 \subset \mathrm{E}_6 \subset \mathrm{E}_7$ action. (The subscript '$\mathcal{L}$' on $\theta_{\mathcal{L}}$ denotes both the use of the imaginary unit l and the identification of the 'leptonic' components of $\theta^1 = \binom{C}{B}$ in $\mathcal{Y}$, as will be seen below. In general the type superscript '1' is not appended to components such as θ_l and $\theta_{\mathcal{L}}$ since they are unambiguously extracted from 'type 1' θ^1 components, while a superscript is included for the 'type 2' or 'type 3' case). Considered as an action of 2×2 matrices $S \in \mathrm{SL}(2,\mathbb{C})^1$ on the 2-component Weyl spinors θ_l and $\theta_{\mathcal{L}}$, extracted from the corresponding θ^1 components of $\mathcal{X}$ and $\mathcal{Y}$ respectively the action of equation 58 may be summarised as:

$$\begin{pmatrix} \theta_l \\ \theta_{\mathcal{L}} \end{pmatrix} \rightarrow \begin{pmatrix} S & 0 \\ 0 & S^{\dagger^{-1}} \end{pmatrix} \begin{pmatrix} \theta_l \\ \theta_{\mathcal{L}} \end{pmatrix} \tag{63}$$

This is precisely the Lorentz transformation of a 4-component Dirac spinor ψ.

As explained in section 5 the components of θ^1 within $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ under the action of $\mathrm{SL}(2,\mathbb{C})^1$ actually decompose into a set of four left-handed Weyl spinors $\{\theta_l, \theta_i, \theta_j, \theta_k\}$ as listed in equation 35. Hence equation 58 contains both the original representation of $\mathrm{SL}(2,\mathbb{C})^1$ on $\mathcal{X}$, which contains the set of four left-handed Weyl spinors in the θ^1 components, simultaneously with an equivalent of the complex conjugate representation on $\mathcal{Y}$, which hence contains a corresponding set of four right-handed Weyl spinors, which may be denoted $\{\theta_{\mathcal{L}}, \theta_I, \theta_J, \theta_K\} \subset \mathcal{Y}$. Correspondingly a set of four 4-component Dirac spinors have hence been identified with:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \theta_l \\ \theta_{\mathcal{L}} \end{pmatrix}, \quad \begin{pmatrix} \theta_i \\ \theta_I \end{pmatrix}, \quad \begin{pmatrix} \theta_j \\ \theta_J \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \theta_k \\ \theta_K \end{pmatrix} \tag{64}$$

each of which transforms in the manner of equation 63 under $S \in \mathrm{SL}(2,\mathbb{C})^1 \subset \mathrm{E}_6 \subset \mathrm{E}_7$ transformations.

The internal $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ symmetry, also described in the previous section, is composed as a subgroup of E_6 purely out of the subset of rotations. Hence these actions are identical on the components of $\mathcal{X}$ and $\mathcal{Y}$ in equation 58. Hence in turn the $\mathrm{SU}(3)_c$ action on the components of $\mathcal{X}$, including upon the θ^1 components as detailed in table 4 and summarised together with the $\mathrm{U}(1)_Q$ action in equation 43, is identical for the corresponding components of $\mathcal{Y}$, and the corresponding $\mathrm{U}(1)_Q$ charges for the respective subcomponents of equation 56 are also the same. Hence the ψ_L and ψ_R components carry matching $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ transformation properties for the set of four Dirac spinors in equation 64 (justifying the identification of both θ_l and $\theta_{\mathcal{L}}$ as leptonic components). Similarly the $\mathrm{SU}(2)^{2,3} \times \mathrm{U}(1)^{2,3} \subset \mathrm{E}_6$ rotations, for the mock electroweak theory described in section 5, also act on the $\mathcal{X}$ and $\mathcal{Y}$ components of $x \in F(\mathrm{h}_3\mathbb{O})$ in the same way.

While the total number of dimensions has been increased from 27 to 56 it remains the case that only a single set of 4 dimensions will describe the external spacetime. This can be chosen as an $\mathrm{h}_2\mathbb{C} \subset \mathrm{h}_3\mathbb{O}$ subset of components $\boldsymbol{v}_4$ within $\mathcal{X}$, under an $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{E}_6$ action, *or* as an $\mathrm{h}_2\mathbb{C} \subset \mathrm{h}_3\mathbb{O}$ subset of components $\boldsymbol{v}_4$ within $\mathcal{Y}$, transforming under the complex conjugate representation, but not *both*. Here we choose $\boldsymbol{v}_4 \equiv \boldsymbol{h}_2 \in \mathrm{h}_2\mathbb{C}$ as embedded within the $Y = \left(\begin{smallmatrix} P & \bar{A} \\ A & M \end{smallmatrix}\right) \in \mathrm{h}_2\mathbb{O}$ components of type 1 in $\mathcal{Y}$ in equation 56 to represent external spacetime, with Lorentz transformations hence described by:

$$\boldsymbol{h}_2 \to \boldsymbol{h}_2' = S^{\dagger^{-1}} \boldsymbol{h}_2 \, S^{-1} \tag{65}$$

rather than equation 18, under the action of $S \in \mathrm{SL}(2,\mathbb{C})^1 \subset \mathrm{E}_6$. The complex subspace with base units $\{1, l\}$ still underlies both the $\mathrm{SL}(2,\mathbb{C})^1$ subgroup and the subspace for the vectors $\boldsymbol{h}_2 \in \mathrm{h}_2\mathbb{C}$. These $\boldsymbol{h}_2$ components of $\mathcal{Y}$ will also now be taken to form the 'vector-Higgs' correlated with the phenomena of the Standard Model Higgs sector and Yukawa couplings, as was described for the original case of $L(\boldsymbol{v}_{27}) = 1$ towards the end of the previous section. Here for the case of $L(\boldsymbol{v}_{56}) = 1$ this now implies that *none* of the 27 components of $\mathcal{X} \in \mathrm{h}_3\mathbb{O} \subset F(\mathrm{h}_3\mathbb{O})$ are identified with components of the external spacetime vectors $\boldsymbol{v}_4 \in TM_4$.

In particular this means that in addition to the d-quark and charged lepton components of left-handed Weyl spinors in $\theta^1 \subset \mathcal{X}$ as identified for equation 43, poten-

tially both u-quark *and* neutral lepton left-handed Weyl spinors might be identified in the X components of $\mathcal{X}$ as described provisionally in the previous section. The $a \in \mathbb{O}$ component of $\mathcal{X}$ has the correct $(0, \frac{2}{3})$ charge structure to describe (ν-lepton, u-quark) particle states, as seen in equations 44 and 45, and is now free to accommodate both states. However while the corresponding imaginary $A(6)$ components of $\mathcal{Y}$ also have an $\dot{S}^1_l$ charge of $\frac{2}{3}$, the $A_{1,l} = (A_1 + A_8 l)$ part of $A \in \mathbb{O}$ in $\mathcal{Y}$ is *occupied* by the above components $\boldsymbol{h}_2 \in \mathrm{h}_2\mathbb{C}$, representing the vector-Higgs and external spacetime, as depicted in equation 66.

$$\begin{pmatrix} \alpha & \left(\begin{array}{c|c} X \sim \theta^1_X \theta^{1\dagger}_X & \theta^1 \\ \hline \theta^{1\dagger} & n \end{array}\right)_{\mathcal{X}} \\ \left(\begin{array}{c|c} Y \sim \theta^1_Y \theta^{1\dagger}_Y & \theta^1 \\ \hline \theta^{1\dagger} & N \end{array}\right)_{\mathcal{Y}} & \beta \end{pmatrix} \sim \begin{pmatrix} & \left(\begin{array}{c|c} `\nu_L\text{'} & e_L \\ `u_L\text{'} & d_L \\ \hline & \end{array}\right)_{\mathcal{X}} \\ \left(\begin{array}{c|c} \boldsymbol{v}_4 \equiv \boldsymbol{h}_2 & e_R \\ `u_R\text{'} & d_R \\ \hline & \end{array}\right)_{\mathcal{Y}} & \end{pmatrix} \tag{66}$$

This in principle provides an explanation for the existence of the left-handed neutrino ν_L while the corresponding right-handed state ν_R is prohibited, at least at the level of the basic symmetry structures, as a feature of the breakdown of left-right symmetry through the identification of external spacetime in the breaking of the full symmetry of $L(\boldsymbol{v}_{56}) = 1$. This observation is accompanied by the caveat concerning the Weyl spinor composition of the components of $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$. With this in mind, and hence with quote marks placed on the ν_L, u_L and u_R states, the relation between the component structure for elements of $x \in F(\mathrm{h}_3\mathbb{O})$, in the form of equations 55 and 56, and the first generation of Standard Model fermions is summarised in equation 66.

Unlike the case for E_6, the Lie algebra E_7 does contain a rank-6 subgroup corresponding to the combined external Lorentz symmetry and internal gauge symmetry of the Standard Model, that is:

$$\mathrm{SL}(2,\mathbb{C}) \;\times\; \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \quad \subset \quad \mathrm{E}_7 \tag{67}$$

Since a similar decomposition cannot be obtained from the E_6 symmetry alone, as described at the end of the previous section, this suggests that the identification of an internal SU(2) symmetry will require the use of some combination of elements from $\phi(C)$ and $\psi(D)$ in equations 60 and 61. Since these additional actions differ on the $\mathcal{X}$ and $\mathcal{Y}$ components an SU(2) identified in this way might be expected to have an asymmetric action on the left and right-handed spinors identified in equation 66.

With the aim of identifying an $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset \mathrm{E}_7$ symmetry action, and in contrast with table 3 in section 5, an internal symmetry might be defined as any group $\underline{G}$ consistent with the subgroup decomposition $\mathrm{SL}(2,\mathbb{C})^1 \times \underline{G} \subset \mathrm{E}_7$ for which the set of $\mathrm{SL}(2,\mathbb{C})^1$ spinors transform under the trivial or fundamental representations of $\underline{G}$. That is, while the external $\mathrm{SL}(2,\mathbb{C})^1$ symmetry partitions the components of $L(\hat{\boldsymbol{v}}) = 1$ into irreducible pieces, including the spinors $\theta_{l,i,j,k}$ of equation 35 and table 2 each

composed of four real components, the internal symmetry $\underline{G}$ respects this partitioning in treating the Weyl spinors as individual components of a representation of $\underline{G}$.

The internal $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ symmetry may also be motivated in this way, as a component of the E_7 decomposition with well-defined representations on the spinor components, as seen in equation 43 for example. In this case the fact that it also *happens* that $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q \subset \mathrm{Stab}_7(TM_4) \subset \mathrm{E}_7$ is responsible for the fact that the gauge bosons associated with QCD and QED *happen* to be massless. At the same time the action of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset \mathrm{E}_7$ might still be expected to be closely related to the subgroups $\mathrm{SU}(2)^{2,3} \times \mathrm{U}(1)^{2,3} \subset \mathrm{E}_6$ acting on the components of $\mathcal{X}$ and $\mathcal{Y}$, which impinge upon the external spacetime TM_4 components, since these actions have several suitable properties in relation to electroweak theory as described for the mock electroweak theory of the previous section.

For the present theory the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry breaks to $\mathrm{U}(1)_Q$ as generated by $\dot{S}^1_l$ of $\mathrm{E}_6 \subset \mathrm{E}_7$, in the Lie algebra, which acts upon $\mathcal{X}$ and $\mathcal{Y}$ in the same way. In the case of the $\mathcal{X}$ components the $S^1_l \equiv \mathrm{U}(1)_Q$ action misses the ν_L components of equation 66 accounting for the charge neutrality of the neutrino. In the case of the $\mathcal{Y}$ components the $S^1_l \equiv \mathrm{U}(1)_Q$ action misses the $\boldsymbol{h}_2$ components of equation 66 here potentially accounting for the massless nature of the photon. These two different aspects of electroweak theory may hence here be described together in terms of the broken E_7 action on the $\mathcal{X}$ and $\mathcal{Y}$ components of $L(\boldsymbol{v}_{56}) = q(x) = 1$ in equation 66.

As an extension of the $L(\boldsymbol{v}_{27}) = 1$ case described in the previous section, mass terms for the fermions and proposed to arise from 'Yukawa like' couplings between the fermion and vector-Higgs $\boldsymbol{v}_4$ components of equation 66, now through the quartic constraint $L(\boldsymbol{v}_{56}) = q(x) = 1$ of equation 57.

The scheme in equation 66 accounts for one family of quarks and leptons with the appropriate transformations under the internal $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ symmetry and external $\mathrm{SL}(2,\mathbb{C})^1$ symmetry. This observation carries the significant caveat that the theory requires further development in order to fully identify u-quark and ν-lepton states that transform appropriately under the external $\mathrm{SL}(2,\mathbb{C})^1$ symmetry. There are several possible mathematical ways in which to decompose the components of $X, Y \in \mathrm{h}_2\mathbb{O}$ into a set of spinors under $\mathrm{SL}(2,\mathbb{C})^1$, such as via the octonion spinor decomposition $X = \theta^1_X \theta^{1\dagger}_X$, or as a sum of such terms, as suggested on the left-hand side of equation 66. However, ideally such a structure should be found to arise naturally within the context of the present theory rather than introduced arbitrarily. In addition the particle states yet to be identified include the second and third generation of fermions, as related through CKM mixing in the case of the quark states, and their relation to the massive gauge bosons associated with electroweak theory in the Standard Model. In the following section we speculate on the possible nature of a yet higher-dimensional form of temporal flow in principle capable of accommodating these phenomena.

7 E_8 Symmetry and the Standard Model

The extension from E_6 acting on $L(\boldsymbol{v}_{27}) = 1$ to E_7 acting on $L(\boldsymbol{v}_{56}) = 1$ can be considered as a continuation of the progression to higher-dimensional forms of temporal

flow which began with the $\mathrm{SL}(2,\mathbb{C})$ Lorentz symmetry of the quadratic form $L(\boldsymbol{v}_4) = 1$ of equation 12, via equation 17, on 4-dimensional spacetime. This progression, as an explicit realisation of the underlying motivation for the present theory as introduced through equations 1 and 2, is summarised here in table 5.

	form	dimensions	space	symmetry	# generators
$L(\boldsymbol{v}_4) = 1$	quadratic	4 spacetime	$\boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C}$	$\mathrm{SL}(2,\mathbb{C})$	6
$L(X) = 1$	quadratic	10 spacetime	$X \in \mathrm{h}_2\mathbb{O}$	$\mathrm{SL}(2,\mathbb{O})$	45
$L(\mathcal{X}) = 1$	cubic	27 temporal	$\mathcal{X} \in \mathrm{h}_3\mathbb{O}$	$\mathrm{E}_{6(-26)}$	78
$L(x) = 1$	quartic	56 temporal	$x \in F(\mathrm{h}_3\mathbb{O})$	$\mathrm{E}_{7(-25)}$	133

Table 5: Four-dimensional spacetime, as a form of temporal flow itself, may be embedded in a progression of higher-dimensional temporal forms.

The highest dimensional form of temporal flow $L(\boldsymbol{v}_{56}) = 1$ has a symmetry breaking pattern to $\mathrm{E}_{6(-26)} \subset \mathrm{E}_{7(-25)}$ with the representations of equation 62 as exhibited by the structure of equation 58. This is analogous to the further breaking pattern of E_6 to $\mathrm{SL}(2,\mathbb{O}) \equiv \mathrm{Spin}^+(1,9)$, as described by the representations of equations 25–27, which is also implied in the structure of left-hand side of equation 66. The $\mathrm{SL}(2,\mathbb{O})$ symmetry of 10-dimensional spacetime is an intermediate stage on the way down to the Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetry which further decomposes the representation space into a Lorentz 4-vector, Weyl spinors and Lorentz scalars, as described in table 2 and now applied to both $\mathcal{X}$ and $\mathcal{Y} \in \mathrm{h}_3\mathbb{O}$, with the external Lorentz 4-vector $\boldsymbol{v}_4 \in TM_4$ accommodated within the $\mathcal{Y}$ components in equation 66, where two further Lorentz scalar components α and β are also identified.

Apart from the three additional scalars, N, α and β in equation 66, the increase in dimension from 27 to 56 does not contain any redundancy in terms of comparison with the structures of the Standard Model. Most of the additional 29 dimensions are interpreted as an augmentation from 2-component Weyl spinors to 4-component Dirac spinors, together with a separation in the identification of the left-handed neutrino state and the external spacetime $\mathrm{h}_2\mathbb{C} \equiv TM_4$ components.

In terms of the dimension of the underlying space, as listed for the sequence of forms $L(\boldsymbol{v}) = 1$ in table 5, we first note that a further expansion from 56 to ~ 80 real components would be sufficient to incorporate Weyl spinors for the ν_L, u_L and u_R states of equations 66. This is deduced by observing that $a \in \mathbb{O}$ of equation 45 has 8 real components while a set of four Weyl spinors requires a total of 16 real components, or alternatively by noting that the decomposition of the form $X = \theta^1_X \theta^{1\,\dagger}_X$, as suggested in the left-hand side of equation 66, involves an augmentation from 10 to 16 real components. With a complete generation of Standard Model fermions then accounted for the second and third generations might also be directly incorporated under a further augmentation from 80 to ~ 240 real components.

Given the progression to larger symmetry groups summarised in table 5 from a mathematical point of view it is also natural to consider whether the Lie group E_8, as the largest exceptional Lie group, represented on a quintic homogeneous form

$L(\boldsymbol{v}) = 1$, may mark one further and final possible step in this sequence. (While we refer to such a hypothetical 'quintic' form, essentially an order greater than quartic is implied). With the smallest non-trivial representation of E_8 being 248-dimensional, this possibility is particularly worth consideration in light of the observations of the previous paragraph. In a similar way that extending the symmetry from E_6 to E_7 led to the incorporation of right-handed as well as left-handed fermion states, ideally a further extension to E_8 would subsume both the E_7 symmetry of the structure in equation 66 and explicitly incorporate also the u-quark and $\nu-$lepton spinor states and a full three generations of fermions all under a higher-dimensional form of $L(\hat{\boldsymbol{v}}) = 1$ with an E_8 symmetry.

The smallest non-trivial representation of E_6 is the **27** which can be expressed as the symmetry of the cubic form $L(\boldsymbol{v}_{27}) = 1$, while for E_7 the **56** representation, again the lowest-dimensional non-trivial representation, can be expressed as the symmetry of the quartic form $L(\boldsymbol{v}_{56}) = 1$. However the **248** representation for E_8 is expressed in terms of the adjoint representation on the 248-dimensional E_8 Lie algebra itself, with no clear interpretation in terms of a symmetry of a form of temporal flow $L(\boldsymbol{v}) = 1$. Indeed the Lie algebra E_8 can be essentially introduced in terms of its action on itself, and constructed in purely *algebraic* terms which may involve the octonions [3], with the absence of any *geometric* motivation or application which might be related to a homogeneous form $L(\boldsymbol{v}) = 1$.

The fact that the smallest non-trivial representation of the 248-dimensional E_8 Lie algebra is expressed as the adjoint representation does not itself preclude the possibility that a **248** representation may *also* be identified in terms of the symmetries of a quintic form $L(\boldsymbol{v}_{248}) = 1$. Given the progression from the cubic polynomial form $\det(\mathcal{X})$ of equations 22 and 23 as an expression of $L(\boldsymbol{v}_{27}) = 1$ with an E_6 symmetry to the terms of the quartic form $q(x)$ of equation 57 underlying the form $L(\boldsymbol{v}_{56}) = 1$ with an E_7 symmetry, a possible quintic form for $L(\boldsymbol{v}_{248}) = 1$ with an E_8 symmetry may be a considerably more complicated mathematical object still. Hence it is perhaps conceivable that such a structure has not been identified through purely algebraic means, even over fifty years after the corresponding E_6 and E_7 structures were first realised [14, 15]. On the other hand if such a mathematical structure does exist, namely a quintic form $L(\boldsymbol{v}_{248}) = 1$ with an E_8 symmetry containing the form $L(\boldsymbol{v}_4)$ with $\mathrm{SL}(2,\mathbb{C})$ symmetry, then as for the other forms of table 5 it *would* naturally apply for the present theory, based on multi-dimensional forms of temporal flow, and further physical consequences would be *expected* to be uncovered in this further progression.

In reference [16], as an example of a more geometrical approach, all of the classical Lie groups are accounted for as isometry groups of bilinear or sesquilinear forms and the first four exceptional Lie groups, G_2, F_4, E_6 and E_7, are described as isometry groups constructed for cubic or quartic forms, but with E_8 essentially absent from the discussion. More generally little reference has been identified in the literature in which a 248-dimensional representation of E_8 is described in terms of an action on a quintic, or any other homogeneous polynomial, form. However in [17, 18] a polynomial of degree *eight* which is invariant as a 248-dimensional representation of the compact real form of E_8 is described, and is closely related to an invariant polynomial for the real form $\mathrm{E}_{8(8)}$.

Considering the possible real forms of E_8 more generally, a suitable candidate

would be $E_{8(-24)}$ since the following maximal subgroups involving the exceptional Lie groups are well known (see for example [19]):

$$\begin{aligned} E_{7(-25)} \times SU(1,1) &\subset E_{8(-24)} \\ E_{6(-26)} \times SO(1,1) &\subset E_{7(-25)} \end{aligned} \tag{68}$$

This suggests the employment of the chain of non-compact real forms $E_{6(-26)} \rightarrow E_{7(-25)} \rightarrow E_{8(-24)}$ as symmetry groups for the corresponding forms of the sequence $L(\boldsymbol{v}_{27}) = 1 \rightarrow L(\boldsymbol{v}_{56}) = 1 \rightarrow L(\boldsymbol{v}_{248}) = 1$, where the first two stages have been described here in sections 4–6, while we are led to the third form as a mathematical prediction of the theory. As for the structure of the first two stages it seems likely that a construction of the final form in this progression will involve the algebraic structure of the octonions in a significant way.

The Lie group generated by the rank-8 E_8 algebra, which is also uniquely the largest exceptional Lie algebra, is large enough to contain a rank-8 decomposition of the form:

$$SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times SU(2) \times U(1) \times U(1) \subset E_8 \tag{69}$$

as can be shown by straightforward analysis of the Dynkin diagrams involved. Hence while the degrees of freedom of the components of $\boldsymbol{v}_{248}$, as an extension from $\boldsymbol{v}_{56} \equiv x \in F(h_3\mathbb{O})$ of equation 66, are sufficient to contain a full three generations of Standard Model fermions and a vector-Higgs, the E_8 symmetry group is comfortably large enough to describe the external Lorentz symmetry together with the internal $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group.

It would be possible to attempt to identify the structures of the Standard Model, as alluded to above, within the components of a quintic form $L(\boldsymbol{v}_{248}) = 1$ with an E_8 symmetry *if* the latter structure was already known and described in the literature. This would continue the approach adopted for the E_6 symmetry of $L(\boldsymbol{v}_{27}) = 1$ and E_7 on $L(\boldsymbol{v}_{56}) = 1$, as based on the corresponding mathematical structures originally discovered in the 1950s [14] and 1960s [15] respectively, for which the consequences of symmetry breaking over M_4 have been studied here in sections 5 and 6.

Alternatively the mathematical structure of E_8 acting on a quintic form underlying $L(\boldsymbol{v}_{248}) = 1$, if it exists, might itself be *constructed* through its application in the present theory as a form of temporal flow based on a knowledge of the empirical properties of the Standard Model. That is, continuing the progression of table 5 through the Standard Model structure identified in the components of $F(h_3\mathbb{O})$ under the broken E_7 symmetry in equation 66, and *using* the need to identify spinor components for the ν-lepton and u-quarks, together with three generations of fermions oriented under an $SU(2)_L$ action and relating to CKM mixing, all in a structural correspondence with the Standard Model, might lead to the identification of a suitable underlying 248-dimensional space. The study of this mathematical structure, incorporating the subspaces of $h_3\mathbb{O}$ and $F(h_3\mathbb{O})$ under the subgroups E_6 and E_7 respectively, may lead to the identification of an E_8 symmetry represented on the form $L(\boldsymbol{v}_{248}) = 1$, which might then be rigorously studied as a mathematical object in its own right, in principle with further consequences for the present theory in turn derived.

8 Conclusions

From the simple starting point of the multi-dimensional form of time, constrained to the form of homogeneous polynomials in the limit of infinitesimal intervals as derived for equation 11, the progression of forms listed in table 5 has been considered. These forms are *not* however restricted to a quadratic spacetime structure, and this has opened up the possibility to explore structures that naturally accommodate a series of Standard Model properties.

The initial extension from a quadratic 4-dimensional spacetime form to a cubic 9-dimensional temporal form, based on elements of $h_3\mathbb{C}$, led immediately to the identification of a left-handed Weyl spinor ψ_L transforming under an internal U(1) symmetry as described for equation 19. On further generalisation to an E_6 symmetry of the 27-dimensional space $h_3\mathbb{O}$ a set of four left-handed spinors under the external Lorentz symmetry was identified, equation 35, which transform under an internal $SU(3)_c \times U(1)_Q$ symmetry as an electron colour singlet and a d-quark colour triplet with the correct relative fractional electromagnetic charge arising directly out of the mathematical structure as described leading up to equation 43. A neutrino and u-quark triplet are also identified provisionally in equation 45 via the properties of further components of $h_3\mathbb{O}$ under the internal $SU(3)_c \times U(1)_Q$ transformations. Structures which closely parallel $SU(2)_L \times U(1)_Y$ electroweak symmetry breaking were also described in section 5. Further extension to an E_7 symmetry on the 56-dimensional space $F(h_3\mathbb{O})$ led to equation 66 with further features of the Standard Model identified, such as the suppression of the right-handed neutrino state and a left-right asymmetry more generally.

All of the above properties have arisen out of the natural mathematical progression of the theory, based on the generalisation from spacetime symmetries as originally introduced for equation 2, through the structures summarised in table 5. At present the main prediction of the theory regards the structure of a currently hypothetical E_8 symmetry action on a quintic or higher polynomial form $L(\boldsymbol{v}_{248}) = 1$ to complete both the progression through the exceptional Lie algebras and the full Standard Model picture. In the meantime the series of Standard Model properties already uncovered makes a strong case for the plausibility of the theory. This theory has also been explored in the directions of Kaluza-Klein geometry, quantisation scheme and applications in cosmology. These developments, together with a broader account of the work presented here, will be detailed in a separate paper, with the ultimate aim of deriving physical predictions both in high energy physics and cosmology.

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A Elements of the E_6 Lie Algebra

$\dot{B}^1_{\underline{tz}}$ $\begin{pmatrix} +p & 0 & +\frac{1}{2}c \\ 0 & -m & -\frac{1}{2}\bar{b} \\ +\frac{1}{2}\bar{c} & -\frac{1}{2}b & 0 \end{pmatrix}$	$\dot{B}^1_{\underline{tx}}$ $\begin{pmatrix} +a_x & \frac{1}{2}(p+m) & +\frac{1}{2}\bar{b} \\ \frac{1}{2}(p+m) & +a_x & +\frac{1}{2}c \\ +\frac{1}{2}b & +\frac{1}{2}\bar{c} & 0 \end{pmatrix}$	$\dot{B}^1_{\underline{tq}}$ $\begin{pmatrix} -a_q & \frac{1}{2}(p+m)q & +\frac{1}{2}q\bar{b} \\ -\frac{1}{2}(p+m)q & -a_q & -\frac{1}{2}qc \\ -\frac{1}{2}bq & +\frac{1}{2}\bar{c}q & 0 \end{pmatrix}$
$\dot{B}^2_{\underline{tz}}$ $\begin{pmatrix} 0 & +\frac{1}{2}\bar{a} & -\frac{1}{2}c \\ +\frac{1}{2}a & +m & 0 \\ -\frac{1}{2}\bar{c} & 0 & -n \end{pmatrix}$	$\dot{B}^2_{\underline{tx}}$ $\begin{pmatrix} 0 & +\frac{1}{2}c & +\frac{1}{2}\bar{a} \\ +\frac{1}{2}\bar{c} & +b_x & \frac{1}{2}(m+n) \\ +\frac{1}{2}a & \frac{1}{2}(m+n) & +b_x \end{pmatrix}$	$\dot{B}^2_{\underline{tq}}$ $\begin{pmatrix} 0 & -\frac{1}{2}cq & +\frac{1}{2}\bar{a}q \\ +\frac{1}{2}q\bar{c} & -b_q & \frac{1}{2}(m+n)q \\ -\frac{1}{2}qa & -\frac{1}{2}(m+n)q & -b_q \end{pmatrix}$
$\dot{B}^3_{\underline{tz}}$ $\begin{pmatrix} -p & -\frac{1}{2}\bar{a} & 0 \\ -\frac{1}{2}a & 0 & +\frac{1}{2}\bar{b} \\ 0 & +\frac{1}{2}b & +n \end{pmatrix}$	$\dot{B}^3_{\underline{tx}}$ $\begin{pmatrix} +c_x & +\frac{1}{2}b & \frac{1}{2}(n+p) \\ +\frac{1}{2}\bar{b} & 0 & +\frac{1}{2}a \\ \frac{1}{2}(n+p) & +\frac{1}{2}\bar{a} & +c_x \end{pmatrix}$	$\dot{B}^3_{\underline{tq}}$ $\begin{pmatrix} -c_q & -\frac{1}{2}qb & -\frac{1}{2}(n+p)q \\ +\frac{1}{2}\bar{b}q & 0 & -\frac{1}{2}aq \\ \frac{1}{2}(n+p)q & +\frac{1}{2}q\bar{a} & -c_q \end{pmatrix}$
$\dot{R}^1_{\underline{xq}}$ $\begin{pmatrix} 0 & -a_q-a_xq & -\frac{1}{2}qc \\ -a_q+a_xq & 0 & +\frac{1}{2}q\bar{b} \\ +\frac{1}{2}\bar{c}q & -\frac{1}{2}bq & 0 \end{pmatrix}$	$\dot{R}^1_{\underline{xz}}$ $\begin{pmatrix} +a_x & -\frac{1}{2}(p-m) & +\frac{1}{2}\bar{b} \\ -\frac{1}{2}(p-m) & -a_x & -\frac{1}{2}c \\ +\frac{1}{2}b & -\frac{1}{2}\bar{c} & 0 \end{pmatrix}$	$\dot{R}^1_{\underline{zq}}$ $\begin{pmatrix} +a_q & \frac{1}{2}(p-m)q & -\frac{1}{2}q\bar{b} \\ -\frac{1}{2}(p-m)q & -a_q & -\frac{1}{2}qc \\ +\frac{1}{2}bq & +\frac{1}{2}\bar{c}q & 0 \end{pmatrix}$
$\dot{R}^2_{\underline{xq}}$ $\begin{pmatrix} 0 & +\frac{1}{2}\bar{a}q & -\frac{1}{2}cq \\ -\frac{1}{2}qa & 0 & -b_q-b_xq \\ +\frac{1}{2}q\bar{c} & -b_q+b_xq & 0 \end{pmatrix}$	$\dot{R}^2_{\underline{xz}}$ $\begin{pmatrix} 0 & +\frac{1}{2}c & -\frac{1}{2}\bar{a} \\ +\frac{1}{2}\bar{c} & +b_x & -\frac{1}{2}(m-n) \\ -\frac{1}{2}a & -\frac{1}{2}(m-n) & -b_x \end{pmatrix}$	$\dot{R}^2_{\underline{zq}}$ $\begin{pmatrix} 0 & +\frac{1}{2}cq & +\frac{1}{2}\bar{a}q \\ -\frac{1}{2}q\bar{c} & +b_q & \frac{1}{2}(m-n)q \\ -\frac{1}{2}qa & -\frac{1}{2}(m-n)q & -b_q \end{pmatrix}$
$\dot{R}^3_{\underline{xq}}$ $\begin{pmatrix} 0 & +\frac{1}{2}q\bar{a} & -c_q+c_xq \\ -\frac{1}{2}aq & 0 & +\frac{1}{2}\bar{b}q \\ -c_q-c_xq & -\frac{1}{2}qb & 0 \end{pmatrix}$	$\dot{R}^3_{\underline{xz}}$ $\begin{pmatrix} -c_x & -\frac{1}{2}b & -\frac{1}{2}(n-p) \\ -\frac{1}{2}\bar{b} & 0 & +\frac{1}{2}a \\ -\frac{1}{2}(n-p) & +\frac{1}{2}\bar{a} & +c_x \end{pmatrix}$	$\dot{R}^3_{\underline{zq}}$ $\begin{pmatrix} -c_q & -\frac{1}{2}qb & -\frac{1}{2}(n-p)q \\ +\frac{1}{2}\bar{b}q & 0 & +\frac{1}{2}aq \\ \frac{1}{2}(n-p)q & -\frac{1}{2}q\bar{a} & +c_q \end{pmatrix}$

Table 6: Vector fields on $Th_3\mathbb{O}$ generated by the 26 Category 1 Boosts and 31 Category 2 Rotations from table 1 in the form of equation 29 ($\dot{B}^3_{\underline{tz}}, \dot{R}^2_{\underline{xq}}, \dot{R}^3_{\underline{xq}}$ are non-basis elements).

		i	j	k	kl	jl	il	l
$\dot{A}_i$:	$\dot{a} =$		$-a_4 j$	$+a_3 k$	$+a_6 kl$	$-a_5 jl$		
$\dot{A}_j$:	$\dot{a} =$	$+a_4 i$		$-a_2 k$	$-a_7 kl$		$+a_5 il$	
$\dot{A}_k$:	$\dot{a} =$	$-a_3 i$	$+a_2 j$			$+a_7 jl$	$-a_6 il$	
$\dot{A}_{kl}$:	$\dot{a} =$	$+a_6 i$	$+a_7 j$			$-a_2 jl$	$-a_3 il$	
$\dot{A}_{jl}$:	$\dot{a} =$	$-a_5 i$		$-a_7 k$	$+a_2 kl$		$+a_4 il$	
$\dot{A}_{il}$:	$\dot{a} =$		$+a_5 j$	$+a_6 k$	$-a_3 kl$	$-a_4 jl$		
$\dot{A}_l$:	$\dot{a} =$	$+a_7 i$	$-a_6 j$			$+a_3 jl$	$-a_2 il$	
$\dot{G}_i$:	$\dot{a} =$		$-a_4 j$	$+a_3 k$	$-a_6 kl$	$+a_5 jl$	$-2a_8 il$	$+2a_7 l$
$\dot{G}_j$:	$\dot{a} =$	$+a_4 i$		$-a_2 k$	$+a_7 kl$	$-2a_8 jl$	$-a_5 il$	$+2a_6 l$
$\dot{G}_k$:	$\dot{a} =$	$-a_3 i$	$+a_2 j$		$-2a_8 kl$	$-a_7 jl$	$+a_6 il$	$+2a_5 l$
$\dot{G}_{kl}$:	$\dot{a} =$	$+a_6 i$	$-a_7 j$	$+2a_8 k$		$-a_2 jl$	$+a_3 il$	$-2a_4 l$
$\dot{G}_{jl}$:	$\dot{a} =$	$-a_5 i$	$+2a_8 j$	$+a_7 k$	$+a_2 kl$		$-a_4 il$	$-2a_3 l$
$\dot{G}_{il}$:	$\dot{a} =$	$+2a_8 i$	$+a_5 j$	$-a_6 k$	$-a_3 kl$	$+a_4 jl$		$-2a_2 l$
$\dot{G}_l$:	$\dot{a} =$	$+a_7 i$	$+a_6 j$	$-2a_5 k$	$+2a_4 kl$	$-a_3 jl$	$-a_2 il$	

$$\dot{S}^1_q : \begin{cases} \dot{a} = & q \sum\limits_{r \neq 1, q} a_r r \\ \dot{b} = & +\frac{3}{2} b_q - \frac{3}{2} b_1 q - \frac{1}{2} q \sum\limits_{r \neq 1, q} b_r r \\ \dot{c} = & -\frac{3}{2} c_q + \frac{3}{2} c_1 q - \frac{1}{2} q \sum\limits_{r \neq 1, q} c_r r \end{cases}$$

$$\dot{S}^2_q : \dot{a} = -\tfrac{3}{2} a_q + \tfrac{3}{2} a_1 q - \tfrac{1}{2} q \sum a_r r, \quad \dot{b} = q \sum b_r r, \quad \dot{c} = +\tfrac{3}{2} c_q - \tfrac{3}{2} c_1 q - \tfrac{1}{2} q \sum c_r r$$

$$\dot{S}^3_q : \dot{a} = +\tfrac{3}{2} a_q - \tfrac{3}{2} a_1 q - \tfrac{1}{2} q \sum a_r r, \quad \dot{b} = -\tfrac{3}{2} b_q + \tfrac{3}{2} b_1 q - \tfrac{1}{2} q \sum b_r r, \quad \dot{c} = q \sum c_r r$$

Table 7: Vector fields on $Th_3\mathbb{O}$ generated by the 21 Category 3 Transverse Rotations from the lower section of table 1. In the case of $\dot{A}_q$ and $\dot{G}_q$ the form of $\dot{b} = f(b)$ and $\dot{c} = f(c)$ is identical to $\dot{a} = f(a)$. With reference to equation 29, in all cases $\dot{p} = \dot{m} = \dot{n} = 0$ with $\{\dot{\bar{a}}, \dot{\bar{b}}, \dot{\bar{c}}\}$ implied from $\{\dot{a}, \dot{b}, \dot{c}\}$. ($\dot{S}^2_q$ and $\dot{S}^3_q$ are non-basis elements, with $\sum := \sum_{r \neq 1, q}$ here).